THE

STATICALLY-INDETERMINATE STRESSES

IN FRAMES COMMONLY USED FOR

BRIDGES

BY

ISAMI HIROI, C. E., Dr. Eg.,
Professor of Civil Engineering in the College of Engineering
Tokyo Imperial University
Author of "Plate-Girder Construction"

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PREFACE TO THE SECOND EDITION

In the present edition, corrections are made of errors which were found in the first issue of the work. The temperature stress in viaduct bents, which the author neglected to work out in the previous edition, is made the subject of an appendix. A proof of the theorems of Castigliano is also appended, forming a supplementary article to the introductory chapter.

I. H.

February, 1915.

PREFACE TO THE FIRST EDITION

THE present work is the outgrowth of a series of lectures given to the students of Civil Engineering in the Tokyo Imperial University. It contains the solution of those problems most commonly met in the practice of a bridge engineer, the aim of the author being to save time and labor of those intent on a more rational design of the class of the structures treated, than is generally followed, by furnishing them with necessary formulas for which rough approximation or even guess-work frequently forms a substitute.

For different treatment of some of the cases discussed in this work, readers may do well to compare the works of Professors Burr, Greene, Du Bois and Johnson, and also those of Professors Engesser, Résal, Winkler, Melan, Müller-Breslau, Steiner, etc.

The author acknowledges his indebtedness for valuable assistance in preparing the volume, to his colaborer Assistant-Professor H. Kimishima.

TOKYO IMPERIAL UNIVERSITY,
August, 1904.

I. H.

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INTRODUCTORY CHAPTER

GENERAL PRINCIPLES

1. Most of the cases of statically-indeterminate stresses occurring in the practice of a bridge engineer can be solved in several different ways; but in this, the author has made the exclusive use of the method of work as the simplest and the most direct way for arriving at the results.

It is a well-known principle in mechanics that when external forces act on an elastic body, the latter undergoes deformations, which, according to the Hooke's law, are proportional to the stresses causing them, — the deformations assumed to be disappearing the moment the forces are taken off. The work thus performed in the body while being acted on by external forces, we call the *work of resistance*. This internal work, which we shall henceforth designate with ω, may be expressed in the following manner, for different kinds of stresses.

2. Direct Stress. — Suppose a straight bar having a cross-section A, length L, and modulus of elasticity E, be subjected to tension or compression increasing from 0 to S. Assuming the strain to be proportional to stress, the bar would undergo, at any moment when the stress is s, a deformation of

$$\frac{sL}{AE}.$$

Since work equals the force into its displacement the increment of work performed in the bar at the moment will be

$$\frac{sL}{AE}ds,$$

so that for the total work of resistance in the bar we get

$$\omega = \int_0^S \frac{sL}{AE}ds = \frac{S^2L}{2AE} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

3. Normal Stress. — If the bar be a curved one with a developed length of L', then representing by N the normal stress acting at any section distant c — measured along the axis of the bar — from one end, we have, by the same reasoning as before, for the work of resistance in the elementary length dc,

$$\frac{N^2dc}{2AE},$$

and for the total internal work in the bar due to N,

$$\omega = \int_0^{L'} \frac{N^2dc}{2AE} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

4. Bending Moment. — Let Fig. 1 represent the portion of a beam, subjected to bending moment M; then in any elementary length dx, at a distance of y from the neutral axis NA, will be found taking place a deformation of

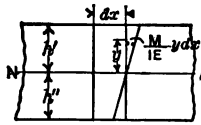

Fig. 1

$$\frac{M}{IE}ydx$$

in the elementary length of the fibre; and at the farthest fibre,

$$\frac{M}{IE}h'dx.$$

Representing by I the moment of inertia of the section, and by b the width of the beam at y, we get for the stress acting in the elementary section bdy, the expression

$$\frac{M}{I}bydy,$$

and consequently, for the work of resistance in the same,

$$\frac{1}{2}\frac{M}{IE}ydx \cdot \frac{M}{I}bydy = \frac{1}{2}\frac{M^2}{I^2E}bdxy^2dy;$$

so that for the total work in the elementary length dx we get

$$\frac{1}{2}\frac{M^2}{I^2E}dx\int_{-h''}^{h'}by^2dy;$$

and as

$$\int_{-h''}^{h'}by^2dy = I,$$

the total work of resistance due to the moment in length l of the beam will be

$$\omega = \int_0^l \frac{M^2dx}{2IE} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

5. Tangential Stress. — The deformation of a beam due to shear is generally so insignificant when compared with that due to the bending, that it may be totally neglected without sensible error in the calculation of internal work. In passing, however, the expression for the work will be given.

Let

T = tangential stress acting at any point distant x from one end of the piece.

G = modulus of elasticity for shear.

A = cross-section of the piece.

Since the action of the tangential stress in the elementary length dx (Fig. 2) is to produce the angular change γ, for which, were T uniformly distributed over the cross-section, we would have,

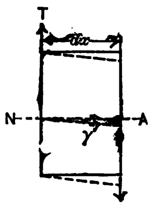

Fig. 2

$$\gamma = \frac{T}{GA},$$

and for the work performed in dx,

$$\tfrac{1}{2} T\gamma dx = \frac{T^2}{2\,GA} dx.$$

Since, however, the intensity of shear at different points of the cross-section differs with the form of the latter, we have for the internal work due to shear,

$$\omega = \int \frac{aT^2 dx}{2\,GA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which

$$a = \frac{A}{T^2}\int_{-h''}^{h'} \tau^2 dA,$$

a quantity always greater than 1.

$$\tau = \frac{T}{bI}\int_{h}^{h'} y dA,$$

h representing the distance of fibres above the neutral axis where τ is to be found, and b, h', h'' and y the same as in Art. 4.

6. Theorems of Castigliano. — The fundamental principle of the method of work has been enunciated by Castigliano in following words: *

I. "*The displacement of the point of application of an*

* "Theorie des Gleichgewichtes elastischer Systeme," von Castigliano.

external force acting on a body — caused by the elastic deformation of the latter — is equal to the first derivative of the work of resistance performed in the body, with respect to the force."

II. "*The partial derivatives of the work of resistance with respect to statically-indeterminate forces which are so chosen that the forces themselves perform no work are equal to zero.*"

In order to make these enunciations clearly understood, an application of the theorems will be made to a simplest case of statically-indeterminate forces. In Fig. 3 let 1 and 2 represent two columns with a length of L, cross-sections of A_1, A_2 and moduli of elasticity of E_1, E_2 conjointly sustaining a load of W. The latter produces reactions

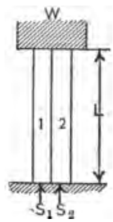

Fig. 3

$$S_1 \text{ and } S_2 = W - S_1,$$

which are at the same time stresses in the columns. Referring to Eq. (1) we get for the internal work in the columns the following expression:

$$\omega = \frac{S_1^2 L}{2 A_1 E_1} + \frac{(W - S_1)^2 L}{2 A_2 E_2}.$$

If we represent by δ the sinking of the load due to compression of the columns, then, according to the first theorem,

$$\frac{d\omega}{dW} = \delta = \frac{(W - S_1) L}{A_2 E_2},$$

and according to the second, since the bases of the columns are assumed to be immovable,

$$\frac{d\omega}{dS_1} = 0 = \frac{S_1 L}{A_1 E_1} + \frac{(S_1 - W) L}{A_2 E_2},$$

from which

$$S_1 = \frac{A_1 E_1}{A_1 E_1 + A_2 E_2} W.$$

7. The second theorem of Castigliano is a direct consequence of the first one, and concerns a special case in which the displacement of the external force is zero. In other words, according to this theorem, a statically-indeterminate force makes the work of resistance a minimum or a maximum. That it is a minimum can be seen by taking the second differential coefficient of ω with respect to the force having a certain amount of displacement. Since the latter will increase with every increment of the force, the second differential will be always positive. For this reason, this theorem is otherwise known as the *principle of least work*, which enunciates that the work of a system of forces acting on an elastic system of construction will be the least possible which is necessary to maintain equilibrium, or, in other words, the external forces so adjust themselves as to develop internal forces in the structure which will make the total work of resistance in the latter a minimum. The principle is a fundamental one in the economy of nature and is applicable to all cases of statically-indeterminate forces in which the forces under question undergo no displacements. For this purpose we have but to express ω in terms of external forces and to differentiate it successively with respect to the forces to be found. The differential coefficients thus obtained, set equal to zero, furnish as many equations

of conditions as there are unknown quantities. The rest of the operation for reduction is a simple algebraic work.

8. It is to be borne in mind that in all forms of structures to be hereafter treated, the joints of every piece, and the piece itself, are assumed to be free from all initial restraints.

CHAPTER I

TRUSSED BEAMS

9. A trussed beam is often treated as a continuous girder resting on fixed supports, and sometimes as so many discontinuous beams as the number of panels into which the beam is divided. That neither treatment is correct hardly requires any explanation.

In the trussed beam of Fig. 4, it is evident that for any load W, as soon as the pressure in the post is made known, stresses in all other members will at once become determinate. Throughout the discussion of beams the following signs will be used:

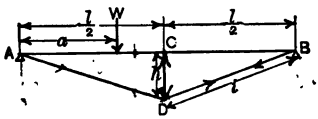

Fig. 4

Compression $-$.

Tension $+$.

Moment $+$, when producing compression on the upper fibres, and vice versâ.

Representing by P the unknown pressure in CD we have the following direct stresses in the several members:

$$\overline{CD} = -P,$$

$$\overline{AD} = \overline{BD} = +\frac{P}{2}\frac{i}{h},$$

$$\overline{AB} = -\frac{P}{2}\frac{l}{2h}.$$

Let A_1, A_2, A_3, represent the cross-sectional areas, and E_1, E_2, E_3, the moduli of elasticity of members AB, CD, and AD respectively. Then referring to Eq. (1) we get for the work of resistance due to the direct stresses the following expressions:

$$\text{Work in } CD \quad . \quad . \quad . \quad . \quad \frac{P^2 h}{2\,E_2 A_2} \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$\text{`` `` } AD \text{ and } BD \quad . \quad \frac{P^2 l^3}{4\,E_3 A_3 h^2} \quad . \quad . \quad . \quad . \quad (b)$$

$$\text{`` `` } AB \quad . \quad . \quad . \quad . \quad \frac{P^2 l^3}{32\,E_1 A_1 h^2} \quad . \quad . \quad . \quad . \quad (c)$$

The beam AB sustains beside the direct stress, the bending moment which at any point distant x from A is

$$A \text{ to } a \quad . \quad . \quad \left\{\frac{W(l-a)}{l} - \frac{P}{2}\right\} x,$$

$$a \text{ to } C \quad . \quad . \quad Wa - \left(\frac{P}{2} + \frac{Wa}{l}\right) x,$$

$$C \text{ to } B \quad . \quad . \quad \left(\frac{Wa}{l} - \frac{P}{2}\right)(l-x),$$

so that for the internal work due to the same we get by referring to Eq. (3) the following expressions:

$$\frac{1}{2\,EI}\left[\int_0^a \left\{\frac{W(l-a)}{l} - \frac{P}{2}\right\}^2 x^2 dx + \int_a^{\frac{l}{2}} \left\{Wa - \left(\frac{P}{2} + \frac{Wa}{l}\right) x\right\}^2 dx \right.$$
$$\left. + \int_{\frac{l}{2}}^{l} \left\{\left(\frac{Wa}{l} - \frac{P}{2}\right)(l-x)\right\}^2 dx\right] \quad . \quad . \quad . \quad (d)$$

in which I denotes the moment of inertia of the section of the beam assumed to be uniform throughout.

Summing up the several works, we get for the total internal work:

$$\omega = (a) + (b) + (c) + (d).$$

Since the value of P must be such as to make ω a minimum, we get for

$$\frac{d\omega}{dP} = 0$$

the following expression,

$$\frac{Ph}{E_2A_2} + \frac{Pi^3}{2\,E_3A_3h^2} + \frac{Pl^3}{16\,E_1A_1h^2} + \frac{1}{2\,E_1I}\Bigg[-\frac{Wa^3(l-a)}{3l} + \frac{Pa^3}{6} + \left\{\frac{P}{2} - \frac{W(l-a)}{l}\right\}\left(\frac{l^3 - 8\,a^3}{24}\right) + \frac{W(l^3 - 8\,a^3)}{24} - \frac{Wa(l^2 - 4\,a^2)}{8} - \frac{Wal^2}{24} + \frac{Pl^3}{48}\Bigg] = 0,$$

from which

$$P = \frac{\dfrac{3\,al^2 - 4\,a^3}{48\,E_1I}}{\dfrac{h}{E_2A_2} + \dfrac{i^3}{2\,h^2E_3A_3} + \dfrac{l^3}{16\,h^2E_1A_1} + \dfrac{l^3}{48\,E_1I}}\,W \qquad (5)$$

To obtain the stress in each member it is simply necessary to substitute this value of P in the expressions for stresses already given. It is evident that the beam AB is subjected to bending and direct stress combined.

Differentiating the second member of Eq. (5) with respect to a and setting the derivative equal to zero, it will be found that P will be maximum for $a = \frac{l}{2}$, as might be anticipated.

10. For a *uniform load* w per unit length we have but to substitute $w\,da$ for W in Eq. (5) and integrate between given limits of loading for each half span. Thus for partial uniform load a_1w (Fig. 5) we get,

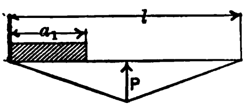

Fig. 5

$$P = \frac{\frac{3l^2 - 2a_1^2}{96 E_1 I} a_1^2 w}{\frac{h}{E_2A_2} + \frac{i^3}{2h^2E_3A_3} + \frac{l^3}{16h^2E_1A_1} + \frac{l^3}{48E_1I}} \quad . \quad . \quad . \quad (6)$$

and for full uniform load wl,

$$P = \frac{\frac{5 l^4 w}{384 E_1 I}}{\frac{h}{E_2A_2} + \frac{i^3}{2h^2E_3A_3} + \frac{l^3}{16h^2E_1A_1} + \frac{l^3}{48E_1I}} \quad . \quad . \quad . \quad (7)$$

EXAMPLE. — A wooden beam 12 in. × 10 in. × 20 feet long between supports, is reinforced by a steel rod 2 in. in diameter and a cast iron strut 3 in. sq. and 2 ft. high. To find the stress in each member due to a full uniform load of 1,200 lbs. per ft. run.

In this case

$$I = \tfrac{1}{12} \times 10 \times 12^3 = 1440$$
$$A_1 = 120 \quad A_2 = 9 \quad A_3 = 3.14 \quad h = 24 \quad l = 240$$
$$i = 122.4 \quad w = 100 \quad \text{(all in in. and lbs.)}$$

Assuming

$E_3 = 30{,}000{,}000$ lbs. per sq. in.
$E_2 = 15{,}000{,}000$ " "
$E_1 = 1{,}500{,}000$ " "

we get in Eq. (7),

$$\frac{5 l^4 w}{384 E_1 I} = 2, \qquad \frac{h}{E_2A_2} = .00000018,$$
$$\frac{i^3}{2h^2E_3A_3} = .000017, \qquad \frac{l^3}{16h^2E_1A_1} = .0000083,$$
$$\frac{l^3}{48E_1I} = .000133,$$

so that

$$P = 12{,}610 \text{ lbs.}$$

Denoting by m the moment at any point x of the beam, we have,

$$m = \frac{24,000 - 12,610}{2} x - \frac{100 x^2}{2}.$$

The maximum moment will be found when $\frac{dm}{dx} = 0$; *i.e.*, for $x = 57$ in., so that

$$\text{max. } m = (5695 - 2850)\, 57 = 162,165 \text{ in. lbs.}$$

The maximum fibre stress in the beam will therefore be

$$\frac{162,165}{1440} \times 6 + \frac{12,610}{2} \times \frac{240}{48} \times \frac{1}{120} = 938 \text{ lbs. per sq. in.}$$

The tension in the tie-rod is equal to

$$\frac{12,610}{2} \times \frac{122.4}{24} \times \frac{1}{3.14} = 10,240 \text{ lbs. per sq. in.}$$

The intensity of compression in the strut CD is simply

$$\frac{12,610}{9} = 1,401 \text{ lbs. per sq. in.}$$

The following table shows the comparison of stresses in the members as calculated above, with those obtained by assuming the beam first to be continuous over three fixed supports and then to consist of two discontinuous beams.

	P in Lbs.	Max. fib. stress in beam, lbs. per sq. in.	Dif.	Tension in tie-rod, lbs. per sq. in.	Dif.
Beam contin. on yield sup.	12,610	938		10,240	
Contin. on 3 fix. sup.	15,000	1,062	+13%	12,180	+19%
Discon. at centre . .	12,000	1,000	+6½%	9,740	− 5%

11. In the trussed beam of Fig. 6 the central panel, owing to its lack of diagonal, is incapable of transmitting shear except through the beam itself. For this reason, so long as the relative positions of points C and E are vertically unchanged (which would practically be the case when the beam AF has sufficient rigidity to remain nearly straight when loaded) the stresses in BC and DE may be

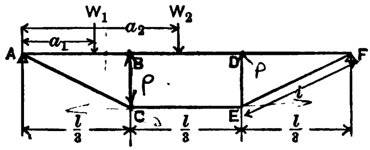

Fig. 6

assumed to be equal. Let P denote the pressure in BC or DE, then by retaining the notations of the preceding case, we obtain the following works of resistance due to any load W, located between A and B:

Work in BC and DE . . . $\dfrac{P^2h}{E_2A_2}$,

" " AC, CE and EF . . $\dfrac{P^2}{E_3A_3h^2}\left(i^3 + \dfrac{l^3}{54}\right)$,

" " AF . . . . . . $\dfrac{P^2l^3}{18\,E_1A_1h^2}$.

The bending moment in the beam causes the following work:

$$\frac{1}{2\,E_1I}\left[\int_0^a\left(\frac{l-a_1}{l}W_1 - P\right)^2 x^2dx + \int_a^{\frac{l}{3}}\left\{\frac{W_1a_1\,(l-x)}{l} - Px\right\}^2 dx\right.$$
$$\left. + \int_{\frac{l}{3}}^{\frac{2l}{3}}\left\{W_1a_1\left(\frac{l-x}{l}\right) - \frac{Pl}{3}\right\}^2 dx + \int_0^{\frac{l}{3}}\left(\frac{W_1a_1}{l} - P\right)^2 x^2dx\right].$$

Summing up these expressions for work and setting the first derivative of the sum with respect to P equal to zero, we get,

$$P\left\{\frac{2h}{E_2A_2}+\frac{2i^3}{E_3A_3h^2}+\frac{l^3}{27\,E_3A_3h^2}+\frac{l^3}{9\,E_1A_1h^2}\right\}$$
$$+\frac{1}{2E_1I}\left[\left\{2P-\frac{2(l-a_1)}{l}W_1\right\}\frac{a_1^3}{3}+\frac{2P}{3}\left(\frac{l^3}{27}-a_1^3\right)\right.$$
$$-\frac{2W_1a_1}{l}\left(\frac{7\,l^3}{162}-\frac{a^2l}{2}+\frac{a^3}{3}\right)+\frac{2Pl^3}{27}-\frac{W_1a_1l^2}{9}$$
$$\left.+\frac{2Pl^3}{81}-\frac{2W_1a_1l^2}{81}\right]=0,$$

from which

$$P=\frac{\dfrac{(2\,l^2-3\,a_1^2)\,a_1}{18\,E_1I}}{\dfrac{2h}{E_2A_2}+\dfrac{2i^3}{E_3A_3h^2}+\dfrac{l^3}{27\,E_3A_3h^2}+\dfrac{l^3}{9\,E_1A_1h^2}+\dfrac{5\,l^3}{81\,E_1I}}W_1. \quad (8)$$

For any load W_2 between B and D we obtain in a similar manner as for W_1 the following expression for P:

$$P=\frac{\dfrac{27\,a_2(l-a_2)-l^2}{162\,E_1I}\,l}{\dfrac{2h}{E_2A_2}+\dfrac{2i^3}{E_3A_3h^2}+\dfrac{l^3}{27\,E_3A_3h^2}+\dfrac{l^3}{9\,E_1A_1h^2}+\dfrac{5\,l^3}{81\,E_1I}}W_2. \quad (9)$$

12. For *full uniform load w* per unit length, by substituting wda for W_1 and W_2 and integrating, we get,

$$P=\frac{\dfrac{11\,l^4w}{486\,E_1I}}{\dfrac{2h}{E_2A_2}+\dfrac{2i^3}{E_3A_3h^2}+\dfrac{l^3}{27\,E_3A_3h^2}+\dfrac{l^3}{9\,E_1A_1h^2}+\dfrac{5\,l^3}{81\,E_1I}}. \quad (10)$$

EXAMPLE. — A beam with dimensions, materials and loading, as in the preceding example, is trussed as in Fig. 6, with $h = 24$ in. $i = 83.5$ in. Then in (10) we have

$$\frac{11\, l^4 w}{486\, E_1 I} = 3.4765, \qquad \frac{2\, l^3}{E_2 A_2 h^2} = .00002145,$$

$$\frac{2\, h}{E_2 A_2} = .00000035, \qquad \frac{l^4}{27\, E_2 A_2 h^2} = .00000922,$$

$$\frac{l^3}{9\, E_1 A_1 h^2} = .00001485, \qquad \frac{5\, l^3}{81\, E_1 I} = .00039506,$$

so that

$$P = 7{,}900 \text{ lbs.}$$

Since the moment at any point x between A and B is

$$\frac{wl - 2\,P}{2}x - \frac{wx^2}{2},$$

the moment will be maximum for $x = 41$, and will be equal to 252,150 in.-lbs.

Again, since at x from A between B and D the moment is

$$\frac{wl - 2\,P}{2}x + P\left(x - \frac{l}{3}\right) - \frac{wx^2}{2},$$

the maximum moment will be found at $x = 119$ in., and will be equal in amount to 87,950 in.-lbs. Taking, then, the first maximum, we get for the maximum fibre stress the following:

$$\frac{252{,}150}{1440} \times 6 + 7900 \times \frac{l}{3\,h} \times \frac{1}{A_2} = 1{,}270 \text{ lbs. per sq. in.}$$

The tension in tie-rods AC and EF equals

$$7900 \times \frac{83.5}{24} \times \frac{1}{3.14} = 8{,}750 \text{ lbs. per sq. in.}$$

while that in tie-rod CE equals

$$7900 \times \frac{80}{24} \times \frac{1}{3.14} = 8{,}060 \text{ lbs. per sq. in.}$$

13. In the two preceding cases of trussed beams, when the depth h of the truss is considerable the influence of the beam will become lost in comparison to the truss, and such a structure approximates itself to a King or Queen post truss whichever it happens to be.

14. A beam reinforced by sloping struts and a straining beam as shown in Fig. 7, is often met with in wooden constructions. The case is somewhat similar to that of Fig. 6. The reinforcing frame *CEFD* could retain its form only when the forces acting at *E* and *F* are equal; and since the beam *AB* will remain practically straight under all circumstances, the reactions produced at *E* and *F* may be assumed to be equal. For any load W_1 between *A* and *E*, then, since Σ vert. forces = o,

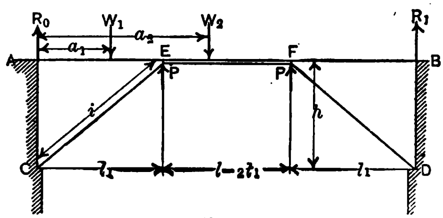

Fig. 7

$$R_0 + R_1 + 2\,P_1 - W_1 = o,$$

in which P_1 denotes that part of P due to W_1. Taking moments successively at B and A, we get,

$$R_0 = \frac{W_1(l - a_1)}{l} - P_1$$

$$R_1 = \frac{W_1 a_1}{l} - P_1.$$

It will be seen from these equations that the reactions at *A* and *B* will, according to modes of loading, be + or −, which latter is to be met by anchoring the beam down to the supports.

The internal work of resistance may now be written,

$$\omega = \frac{P_1^2}{EA}\,\frac{i^3}{h^2} + \frac{P_1^2 l_1^2(l-2l_1)}{2EAh^2} + \int_0^l \frac{m^2dx}{2EI},$$

in which A represents the sectional area of individual members of the frame, I the moment of inertia of the beam AB, and m the moment at any point x of the beam. Substituting in this the following expressions for m (the origin of x being taken at A),

A to a_1 . . $\left\{\frac{W_1(l-a_1)}{l} - P_1\right\}x,$

a_1 to E . . $\frac{W_1a_1(l-x)}{l} - P_1x,$

E to F . . $\frac{W_1a_1(l-x)}{l} - P_1l_1,$

B to F . . $\left(\frac{W_1a_1}{l} - P_1\right)(l-x),$

and differentiating ω with respect to P_1 and setting the differential coefficient equal to zero, we get,

$$P_1 = \frac{\frac{a_1}{6I}(3l_1l - 3l_1^2 - a_1^2)}{\frac{2i^3}{Ah^2} + \frac{l_1^2(l-2l_1)}{Ah^2} + \frac{l_1^2(3l-4l_1)}{3I}}W_1 \quad . \quad . \quad (11)$$

For any load W_2 between E and F similarly we get

$$P_2 = \frac{\frac{l_1}{6I}(3a_2l - l_1^2 - 3a_2^2)}{\frac{2i^3}{Ah^2} + \frac{l_1^2(l-2l_1)}{Ah^2} + \frac{l_1^2(3l-4l_1)}{3I}}W_2 \quad . \quad . \quad (12)$$

It is evident that in this kind of construction, the beam AB may be considered to be free of direct stress.

EXAMPLE. — A wooden beam 12 in. × 8 in. × 30 ft. is reinforced by sloping struts and a straining beam 8 in. × 8 in., with $l_1 = 10$ ft. and $h = 8$ ft. To find the maximum stress in each member due to a full uniform load of 1800 lbs. per ft. run.

Substituting in Eqs. (11) and (12) the following values,

$$i = 12.8 \text{ ft.}, \quad A = 64 \text{ sq. in.}, \quad I = 1152 \text{ in.}^4,$$
$$\frac{2\,i^3}{Ah^2} = 12, \quad \frac{l_1^2(l - 2\,l_1)}{Ah^2} = 3, \quad \frac{l_1^2(3\,l - 4\,l_1)}{3\,I} = 2500,$$
$$6\,I = 6912,$$

we get,

$$P_1 = \frac{2\int_0^{l_1} a\,(3\,l_1 l - 3\,l_1^2 - a^2)\,w da}{6912 \times 2515} = \frac{2 \times \frac{11}{324} l^4 w}{6912 \times 2515},$$

$$P_2 = \frac{\int_{l_1}^{2\,l_1} l_1\,(3\,al - l_1^2 - 3\,a^2)\,wda}{6912 \times 2515} = \frac{\frac{11}{162} l^4 w}{6912 \times 2515},$$

and consequently

$$P = P_1 + P_2 = \frac{(2 \times 27{,}500 + 55{,}000)\,12^4 \times 150}{6912 \times 2515} = 19{,}680 \text{ lbs.}$$

Comparing the maximum moments in the side and central panels, it will be seen that in this case the greatest moment is found at E and F and is equal in amount to

$$\frac{30 \times 1800 - 19{,}680 \times 2}{2} \times 10 - \frac{1800 \times 10^2}{2} = -\,16{,}800 \text{ ft.-lbs.},$$

so that the maximum fibre stress in the beam will be

$$\frac{16{,}800 \times 12}{1152} \times 6 = 1050 \text{ lbs. per sq. in.}$$

Other stresses are as follows:

Sloping strut, $19{,}680 \times \frac{12.8}{8} \times \frac{1}{64} = 492$ lbs. per sq. in.

Straining beam, $19{,}680 \times \frac{10}{8} \times \frac{1}{64} = 345$ lbs. per sq. in.

15. In wooden beam-bridges, the supports, instead of being of masonry, often consist of pile-works such as shown in Fig. 8. In such a construction the struts move laterally, owing to the bending of the piles. It is necessary, therefore, in the summation of internal works, to take in the work in the posts, each of which having one end firmly fixed is simply supported at the other with a horizontal load of $\frac{Pl_1}{h}$ acting at C and D. Calling the moment of inertia of the pile I_0 and neglecting the influence of direct stress as being inconsiderable when compared with that of moment, we get the following internal work in the two posts due to $\frac{Pl_1}{h}$, the dotted portion of the structure not being considered,

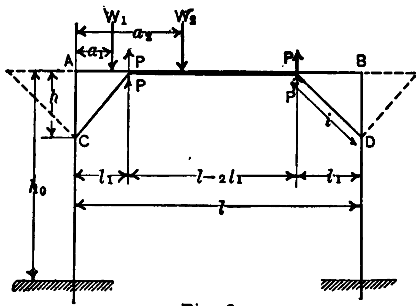

Fig. 8

$$\frac{1}{EI_0}\left[\int_0^h\left(\frac{M}{h_0} - \frac{Pl_1}{h}\frac{h_0 - h}{h_0}\right)^2 x^2 dx + \int_h^{h_0}\left(\frac{M}{h_0}x - Pl_1\frac{h_0 - x}{h_0}\right)^2 dx\right],$$

in which M represents the moment at the base of the post, and x the distance from A downward. Since the value of M must be such as to make this work a minimum, setting the first derivative of the above expression with respect to M equal to zero, we get,

$$M = \frac{(h_0^2 - h^2)}{2\,h_0^2} P l_1.$$

Substituting this value of M in the above expression, the latter becomes,

$$\frac{P^2 l_1^2}{EI_0} \left\{ \frac{(h_0 - h)^3 (3\,h_0 + h)}{12\,h_0^3} \right\}.$$

Adding this to the total work of the preceding case (Art. 14), and differentiating with respect to P and setting the differential coefficient equal to zero, we obtain the following equations for loads W_1 and W_2 corresponding to Eqs. (11) and (12):

$$P_1 = \frac{\dfrac{a_1}{6\,I}(3\,l_1 l - 3\,l_1^2 - a_1^2)}{\dfrac{2\,i^3}{Ah^2} + \dfrac{l_1^2(l-2\,l_1)}{Ah^2} + \dfrac{l_1^2(3\,l-4\,l_1)}{3\,I} + \dfrac{l_1^2(h_0-h)^3(3\,h_0+h)}{6\,h_0^3 I_0}} W_1. \quad (13)$$

$$P_2 = \frac{\dfrac{l_1}{6\,I}(3\,a_2 l - l_1^2 - 3\,a_2^2)}{\dfrac{2\,i^3}{Ah^2} + \dfrac{l_1^2(l-2\,l_1)}{Ah^2} + \dfrac{l_1^2(3\,l-4\,l_1)}{3\,I} + \dfrac{l_1^2(h_0-h)^3(3\,h_0+h)}{6\,h_0^3 I_0}} W_2. \quad (14)$$

The actual measure of h_0 will always be a matter of judgment according to the nature of the ground into which the piles are driven. The heads of piles may in most cases be assumed to be laterally fixed in position.

When there are several consecutive spans with sloping struts as shown dotted and in full, the thrusts at C or D due to full uniform load will balance each other, and the case will be that of Fig. 7, while load on one span only will produce action intermediate between the cases of Figs. 7 and 8.

EXAMPLE. — Using the same dimensions and load as in the preceding example (Art. 14) and further with

$$h_0 = 18 \text{ ft.} \qquad I_0 = \tfrac{1}{12} \times 15^3 \times 15 = 4220 \text{ in.}^4$$

to find the stresses in different members of the frame.

Here we have as before,

$$\frac{2\,i^3}{Ah^2} = 12. \qquad \frac{l_1^2(l - 2\,l_1)}{Ah^2} = 3, \qquad \frac{l_1^2(3\,l - 4\,l_1)}{3\,I} = 2500,$$

$$\frac{l_1^2(h_0 - h)^3(3\,h_0 + h)}{6\,h_0^3 I_0} = 72, \qquad 6\,I = 6912,$$

so that

$$P = P_1 + P_2 = \frac{2 \times 27{,}500 + 55{,}000}{6912\,(2515 + 72)} \times 12^4 \times 150 = 19{,}130 \text{ lbs.}$$

Since the moment at any point x from the end of the beam in the side span is, in this case,

$$\left(\frac{1800 \times 30}{2} - 19{,}130\right)x - \frac{1800\,x^2}{2},$$

it will be maximum for $x = 4.36$ ft., and as it is found to be greater than the maximum moment in the central span, the maximum moment in the beam will be

$$4.36\,(7870 - 900 \times 4.36) = 17{,}200 \text{ ft.-lbs.}$$

The beam acting as a tie for post-heads will have to resist a pull of

$$\frac{Pl_1}{h}\left(\frac{h_0 - h}{h_0} - \frac{h\,(h_0^2 - h^2)}{2\,h_0^3}\right) = 9020 \text{ lbs.}$$

We now have the following intensities of stress:

Beam . . . $\frac{17{,}200 \times 12}{1152} \times 6 + \frac{9020}{96} = 1169$ lbs. per sq. in.

Sloping strut . $19{,}130 \times \frac{12.8}{8} \times \frac{1}{64} = 478$ lbs. per sq. in.

Straining beam, $19{,}130 \times \frac{10}{8} \times \frac{1}{64} = 373$ lbs. per sq. in.

In the post, comparing the moments at C with M, the latter is found to be the greater, and we obtain for the maximum fibre stress,

$$\frac{27{,}000}{15 \times 15} + \frac{76{,}756 \times 12}{4220} \times 6 = 1429 \text{ lbs. per sq. in.},$$

showing that a considerable stress is thrown into the post in such a construction.

CHAPTER II

VIADUCT BENTS

16. FIG. 9 shows a common form of bents of an elevated railway, the posts being riveted to the cross-girder on top and firmly anchored at the base. In this kind of construction the bending of the cross-girder due to loading is transmitted to posts, so that the latter are subjected to combined stresses.

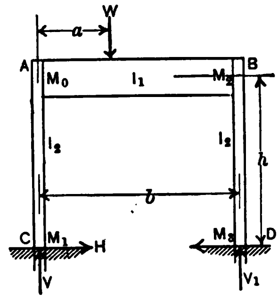

Fig. 9

The following designations will be used:

M_0, M_1, M_2, and M_3 . .	moments at points A, C, B, and D respectively.
I_1 and I_2	the moments of inertia of sections of cross-girder and posts respectively.
h	the height of neutral axis of the cross-girder above the plane of anchorage.
b	the distance apart of the axes of the posts.
V and V_1	the vertical reactions, positive upward, at C and D respectively.
H	the horizontal reaction at C or D, positive when directed toward right.
E	the modulus of elasticity assumed to be constant.

Calling those moments producing compression on the outside fibre of the structure *positive*, we have the following moments:

$M_1 - Hx$ . . . . in post CA at x from C.
$M_1 + Vx - Hh$ . . . . . . in cross-girder at x from A between A and W.
$M_1 + Vx - W(x - a) - Hh$ . . in cross-girder at x from A between W and B.
$M_3 - H(h - x)$ . . . . . . in post BD at x from B.

The total internal work of resistance in the members composing the bent, if we neglect the effect of all direct stresses as being inconsiderable when compared with that of the moment, would be,

$$\omega = \frac{1}{2EI_2}\left\{\int_0^h (M_1 - Hx)^2 dx + \int_0^h (M_3 - Hh + Hx)^2 dx\right\} + \frac{1}{2EI_1}\left[\int_0^a (M_1 + Vx - Hh)^2 dx + \int_a^b \{M_1 + Vx - W(x-a) - Hh\}^2 dx\right].$$

Noting that

$$V = \frac{W(b-a) + M_3 - M_1}{b}$$

the first derivatives of ω taken with respect to M_1, M_3, and H successively set equal to zero, will give the following equations of conditions:

$$\frac{d\omega}{dM_1} = M_1\left(\frac{h}{I_2} + \frac{b}{3I_1}\right) + M_3\frac{b}{6I_1} - Hh\left(\frac{h}{2I_2} + \frac{b}{2I_1}\right) + \frac{Wa}{6bI_1}(2b - a)(b - a) = 0.$$

$$\frac{d\omega}{dM_3} = M_3\left(\frac{h}{I_2} + \frac{b}{3I_1}\right) + M_1\frac{b}{6I_1} - Hh\left(\frac{h}{2I_2} + \frac{b}{2I_1}\right) + \frac{Wa}{6bI_1}(b^2 - a^2) = 0.$$

$$\frac{d\omega}{dH} = (M_1 + M_3)\left(\frac{h}{2\,I_2} + \frac{b}{2\,I_1}\right) - Hh\left(\frac{2\,h}{3\,I_2} + \frac{b}{I_1}\right) + \frac{Wa}{2\,I_1}(b-a) = 0.$$

Combining these equations, we get the following values of H, M_1, and M_3:

$$H = \frac{3\,I_2 a\,(b-a)}{2\,h\,(hI_1 + 2\,bI_2)}W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$M_1 = \frac{I_2}{2}\left\{\frac{1}{hI_1 + 2\,bI_2} - \frac{b - 2\,a}{b\,(6\,hI_1 + bI_2)}\right\}a\,(b-a)\,W \quad . \quad (16)$$

$$M_3 = \frac{I_2}{2}\left\{\frac{1}{hI_1 + 2\,bI_2} + \frac{b - 2\,a}{b\,(6\,hI_1 + bI_2)}\right\}a\,(b-a)\,W \quad . \quad (17)$$

Since

$$M_0 = M_1 - Hh \text{ and } M_2 = M_3 - Hh,$$

we get,

$$M_0 = -\frac{I_2}{2}\left\{\frac{2}{hI_1 + 2\,bI_2} + \frac{b - 2\,a}{b\,(6\,hI_1 + bI_2)}\right\}a\,(b-a)\,W. \quad (18)$$

$$M_2 = -\frac{I_2}{2}\left\{\frac{2}{hI_1 + 2\,bI_2} - \frac{b - 2\,a}{b\,(6\,hI_1 + bI_2)}\right\}a\,(b-a)\,W. \quad (19)$$

Again, since

$$V = \frac{1}{b}\{W\,(b-a) + M_3 - M_1\},$$

we now get

$$V = \left\{1 + \frac{I_2\,a\,(b - 2\,a)}{b\,(6\,hI_1 + bI_2)}\right\}\frac{b-a}{b}W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

It is evident that the maximum stress in post AC is caused by the combined action of V and the moment M_1 or M_0, whichever is greater.

Similarly, since

$$V_1 = \frac{1}{b}\{Wa + M_1 - M_3\},$$

we get

$$V_1 = \left\{1 - \frac{I_2\,(b-a)\,(b - 2\,a)}{b\,(6\,hI_1 + bI_2)}\right\}\frac{a}{b}W.$$

17. If we represent by h_0 and h_1 the distances of the points of contraflexure in the posts (Fig. 10), since

$$M_1 - Hh_0 = 0,$$
$$M_3 - Hh_1 = 0,$$

we get from the preceding equations,

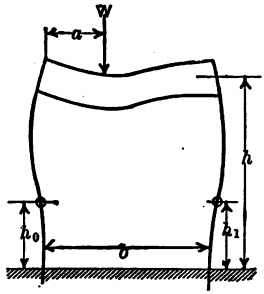

Fig. 10

$$h_0 = \frac{hI_1 + 2\,bI_2}{3}\left\{\frac{1}{hI_1 + 2\,bI_2} - \frac{b - 2\,a}{b\,(6\,hI_1 + bI_2)}\right\}h \quad . \quad . \quad (21)$$

$$h_1 = \frac{hI_1 + 2\,bI_2}{3}\left\{\frac{1}{hI_1 + 2\,bI_2} + \frac{b - 2\,a}{b\,(6\,hI_1 + bI_2)}\right\}h \quad . \quad . \quad (22)$$

For $a = \frac{b}{2}$, then,

$$h_0 = h_1 = \frac{1}{3}h,$$

showing that M_0 and M_2 will then be opposite and twice in amount of M_1 and M_3 respectively.

18. The same condition will obtain in case of loads symmetrically disposed with respect to the centre of the cross-girder. Thus in the case of a single-track bent (Fig. 11), Eqs. (16) to (19) will give for a loading of 2 W,

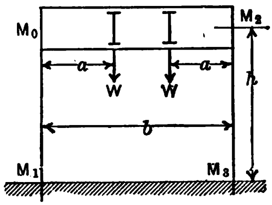

Fig. 11

$$M_0 = M_2 = -\frac{2\,I_2\,a\,(b - a)}{hI_1 + 2\,bI_2}W \quad . \quad . \quad . \quad . \quad (23)$$

$$M_1 = M_3 = \frac{I_2 a\,(b - a)}{hI_1 + 2\,bI_2}W \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

19. In case the posts are *hinged* at C and D (Fig. 12), M_1 and M_3 will disappear from the preceding equations, and

$$H = \frac{3 I_2 a (b - a)}{2 h (2 hI_1 + 3 bI_2)} W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Consequently,

$$M_0 = - Hh = - \frac{3 I_2 a (b - a)}{2 (2 hI_1 + 3 bI_2)} W . \quad . \quad . \quad (26)$$

If the posts were hinged at A and B, there would of course be no moment in the posts.

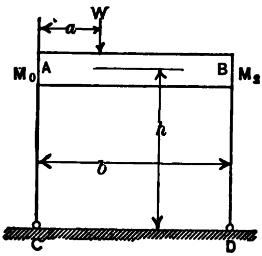

Fig. 12

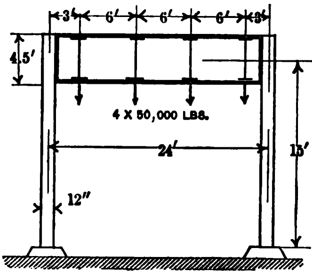

Fig. 13

Example. — In the elevated-railway bent of Fig. 13 to calculate the maximum stress in posts under full loading. Given,

$$I_1 = 24{,}000 \text{ in.}^4$$
$$I_2 = 1\,000 \text{ in.}^4$$
$$A_2 = 24 \text{ ins.}^2$$
$$W = 50{,}000 \text{ lbs.}$$

From Eqs. (23) and (24),

$$M_0 = - \frac{2 I_2 W}{hI_1 + 2 bI_2} \sum_{0}^{\frac{b}{2}} a (b - a) = - 582{,}350 \text{ in.-lbs.}$$

$$M_1 = \frac{I_2 W}{hI_1 + 2bI_2} \sum_0^b a(b-a) = 291{,}170 \text{ in.-lbs.}$$

$$V = \sum_0^b \frac{b-a}{b} W = 100{,}000 \text{ lbs.}$$

The maximum stress in the post will then be,

$$\frac{582{,}350}{1000} \times 6 + \frac{100{,}000}{24} = 7660 \text{ lbs. per sq. in.}$$

Stresses due to moment in the longitudinal plane and those due to changes of temperature remain still to be provided for in the posts.

In case the lower end of each post is hinged, we get from Eq. (26),

$$M_0 = -\frac{3\,I_2 W}{2(2\,hI_1 + 3\,b\,I_2)} \sum_0^b a(b-a) = -\ 450{,}000 \text{ in.-lbs.},$$

so that the maximum stress in the post will be,

$$\frac{450{,}000}{1000} \times 6 + \frac{100{,}000}{24} = 6870 \text{ lbs. per sq. in.}$$

It will thus be seen that so far as the vertical loading is concerned, the stress in the post is increased by fixing the lower ends of the posts.

20. Wind pressure also produces moments in the posts and cross-girder of a bent. In the bent of Fig. 14, in which the posts are constrained at both ends, the wind pressures P and P_1, assumed to be acting as shown by arrows in the axis of the cross-girder, tend to deform the bent, as shown exaggerated in the figure. With the symmetrical disposition of materials and end conditions, as is the case under consideration, it will be easy to see, without going into analytical works, that so long as we do not take into consideration the lengthwise deformation of the girder AB, which is generally inconsiderable when compared with the deflection of the posts, the relative position of points A and B would always remain unchanged, and as a consequence both posts would be equally bent by P and P_1, or, in other words, the reactions H and H_1 would be equal, and the points of contraflexure in the posts would be at the same height above C and D. Then, since equilibrium requires that

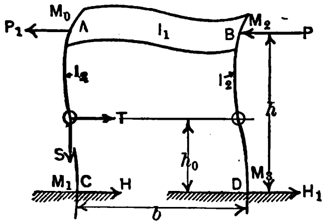

Fig. 14

$$P_1 + P - (H + H_1) = 0,$$

we may put

$$H = H_1 = \frac{P + P_1}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Passing a section through one of the points of contraflexure, and representing by S and T the direct and tangential stresses at the section of the upper portion (the opposite stresses of equal amount being assumed to be acting at the section of the lower portion), we have

$$H = T,$$

and taking moment at C (moments + when producing compression on the outside fibre as before),

$$M_1 = Hh_0 .$$

In the upper side of the section, taking moments successively at A, o, B and D, we get,

$$M_0 = -H(h - h_0).$$
$$Sb = -(P_1 + P)(h - h_0).$$
$$M_2 = -T(h - h_0) - Sb = H(h - h_0) = -M_0.$$
$$M_3 = Th_0 - Sb - (P + P_1)h = -Hh_0 = -M_1.$$

The moment at any point of the frame may now be expressed as follows, the origin of x being taken at A, C, and D respectively:

Cross-girder, $M_0 + \frac{M_2 - M_0}{b}x = -H(h - h_0) + \frac{2H(h - h_0)}{b}x.$

Left post, $M_1 + \frac{M_0 - M_1}{h}x = H(h_0 - x).$

Right post, $M_3 + \frac{M_2 - M_3}{h}x = H(-h_0 + x).$

Neglecting the influence of direct stresses and shears as before, we obtain for the internal work:

$$\omega = \frac{H^2}{2EI_1}\int_0^b (h - h_0)^2\left(\frac{2x}{b} - 1\right)^2 dx + \frac{H^2}{EI_2}\int_0^h (h_0 - x)^2\,dx.$$

Putting the first derivative of ω with respect to h_0, equal to zero, we obtain,

$$h_0 = \frac{bI_2 + 3hI_1}{bI_2 + 6hI_1}h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and consequently,

$$M_0 = -M_2 = -H(h - h_0) = -\frac{3hI_1}{bI_2 + 6hI_1}hH \quad . \quad (29)$$

$$M_1 = -M_3 = Hh_0 = \frac{bI_2 + 3hI_1}{bI_2 + 6hI_1}hH \quad . \quad . \quad . \quad . \quad (30)$$

The *direct stress* in the post is S, being compression in AC and tension in BD.

$$S = \frac{(P_1 + P)(h - h_0)}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The posts are, therefore, subjected to bending and direct stress combined.

If the posts were *hinged* at the base, M_1 and M_3 would disappear, and as h_0 would then be equal to o, we get

$$M_0 = - M_2 = - Hh.$$

The reverse actions would take place if the girder were hinged to the posts while the latter are fixed at the base.

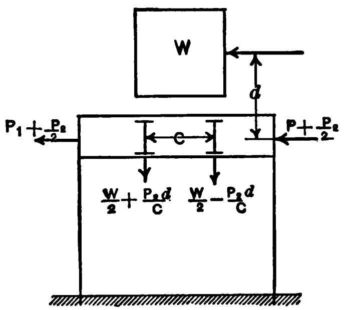

Fig. 15

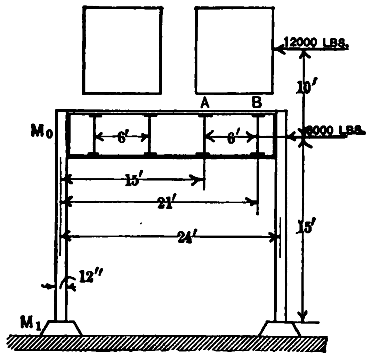

Fig. 16

21. In case the wind pressure P_2 acts on the train at a distance of d (Fig. 15) above the axis of the cross-girder in addition to P and P_1 acting on the structure as before, the bent will beside its own weight be subjected to forces due to the overturning moment of P_2, as shown in the figure. The moments as found by Eqs. (16) to (19) for vertical forces are in this case to be combined with those by Eqs. (29) and (30) for horizontal forces to obtain the resultant moments and reactions.

EXAMPLE.—In the bent of the preceding example, suppose wind pressures of 12,000 lbs. and 6000 lbs. be acting on the exposed surfaces of the train and viaduct respectively (Fig. 16).

To find the stresses produced in the posts due to these wind pressures only.

Considering the horizontal forces only, a pressure of 18,000 lbs. acting at the axis of the cross-girder will cause in lee-side post, according to Eqs. (29) and (30), the following moments:

$$M_0 = -\frac{45 \times 24{,}000}{24 \times 1000 + 90 \times 24{,}000} \times 15 \times \frac{18{,}000}{2} = -66{,}780 \text{ ft.-lbs.}$$

$$M_1 = \frac{24 \times 1000 + 45 \times 24{,}000}{24 \times 1000 + 90 \times 24{,}000} \times 15 \times \frac{18{,}000}{2} = 68{,}240 \text{ ft.-lbs.}$$

The overturning moment of wind pressure on the train produces one upward and another downward pressure of $\frac{12{,}000 \times 10}{6} = 20{,}000$ lbs. on the cross-girder at A and B, for which we get, from Eqs. (18) and (16),

$$M_0 = -\frac{1000}{2} \times 20{,}000 \left\{ \left(\frac{2}{408{,}000} + \frac{24 - 30}{52{,}416{,}000} \right) 15 \times 9 - \left(\frac{2}{408{,}000} + \frac{24 - 42}{52{,}416{,}000} \right) 21 \times 3 \right\} = -3580 \text{ ft.-lbs.}$$

$$M_1 = \frac{1000}{2} \times 20{,}000 \left\{ \left(\frac{1}{408{,}000} - \frac{24 - 30}{52{,}416{,}000} \right) 15 \times 9 - \left(\frac{1}{408{,}000} - \frac{24 - 42}{52{,}416{,}000} \right) 21 \times 3 \right\} = 1700 \text{ ft.-lbs.}$$

As to the direct stress in the posts, we get, from Eqs. (31) and (20),

$$V = \frac{18{,}000\,(h - h_0)}{24} + 5000 = 10{,}565 \text{ lbs.}$$

The maximum fibre stress in the left post due to wind pressure will therefore be a compression of

$$\frac{(66{,}780 + 3580)\ 12}{1000} \times 6 + \frac{10{,}565}{24} = 5505 \text{ lbs. per sq. in.}$$

This amount of fibre stress is but little less than that due to full loading found in the preceding example.

In case the lower ends of the posts are hinged, M_0 would be

$$-\ 9000 \times 15 = -\ 135{,}000 \text{ ft.-lbs. nearly,}$$

and

$$V = \frac{12{,}000 \times 25 + 6000 \times 15}{24} = 16{,}250 \text{ lbs.,}$$

so that the maximum fibre stress will not be less than

$$\frac{135{,}000 \times 12 \times 6}{1000} + \frac{16{,}250}{24} = 10{,}397 \text{ lbs. per sq. in.}$$

These figures show that the decrease of moment due to vertical loading by hinging the lower ends of the posts is far more than neutralized by the increased moment caused in the same by wind pressure.

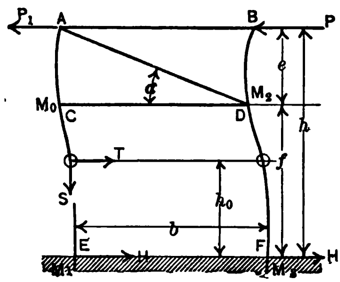

Fig. 17

22. In a bent with simple cross-bracing, such as shown in Fig. 17, acted on by wind pressures P and P_1, if we suppose all the joints of the bracing to be hinged, there would be two points of no moment in each post, viz.: A,B, and the points of contraflexure o, o. Also there would be no moment in any member of the bracing. Passing a section through o of the post AE, and denoting by T and S the tangential and direct stresses acting at the section of the upper portion, as in the preceding case, we have

$$T = H = \frac{P + P_1}{2}.$$

Taking moments at E and C,

$$M_1 = Hh_0,$$
$$M_0 = - T(f - h_0) = - H(f - h_0),$$
$$M_2 = - M_0,$$
$$M_3 = - M_1.$$

Taking moment at o of the post BF,

$$- Sb - (P_1 + P)(h - h_0) = 0,$$

or

$$S = - \frac{(P_1 + P)(h - h_0)}{b} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Then at any point distant x from E we have the following moments in the post AE:

E to C, $\quad M_1 + \frac{M_0 - M_1}{f} x = H(h_0 - x).$

C to A, $\quad M_0 - \frac{M_0}{e}(x - f) = H(f - h_0)\left(\frac{x - f}{e} - 1\right).$

The corresponding moments in the post BF have simply the opposite signs.

Neglecting, then, the influence of all direct and tangential stresses both in the posts and bracing, we get for the internal work of moments in the posts:

$$\omega = \frac{H^2}{EI}\left\{\int_0^f (h_0 - x)^2\,dx + \int_f^h (f - h_0)^2\left(\frac{x-f}{e} - 1\right)^2 dx\right\}.$$

Integrating, we obtain,

$$\omega = \frac{H^2}{EI}\left\{f\left(h_0^2 - h_0 f + \frac{f^2}{3}\right) + \frac{e}{3}(f - h_0)^2\right\}.$$

Differentiating ω with respect to h_0, and setting the differential coefficient equal to zero, we get

$$h_0 = \frac{f(2h + f)}{2(h + 2f)} \quad \ldots\ldots \quad (33)$$

so that

$$M_1 = -M_3 = \frac{f(2h+f)}{2(h+2f)} H \quad \ldots\ldots \quad (34)$$

$$M_0 = -M_2 = -\frac{3f^2}{2(h+2f)} H \quad \ldots\ldots \quad (35)$$

To obtain direct stresses in the posts and braces, pass a section through *AB*, *AD*, and *CD*; then, considering the left portion of the section as far as to the point of contraflexure, the moment taken with respect to *D* will give, by calling, as before, compression — and tension +:

$$-Sb - T(f - h_0) - P_1 e + \overline{AB}\cdot e = 0,$$

from which

$$\overline{AB} = \frac{Sb + T(f - h_0) + P_1 e}{e} = -(P + P_1)\frac{(h - h_0)}{2e} + \frac{P_1 - P}{2}.$$

Taking moment at A, we get

$$- T (h - h_0) - \overline{CD} \cdot e = 0,$$

from which

$$\overline{CD} = - \frac{T (h - h_0)}{e} = - \frac{(P + P_1)(h - h_0)}{2e}.$$

Further, since $\Sigma V = 0$ at the section,

$$- S - \overline{AD} \sin \alpha = 0.$$

Whence

$$\overline{AD} = - \frac{S}{\sin \alpha} = \frac{(P_1 + P)(h - h_0)}{b \sin \alpha}.$$

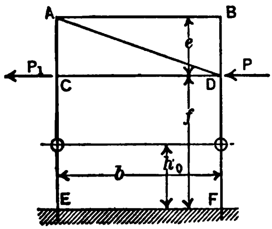

Fig 18

The direct stress in either post is nothing else than S, being compression in AE and tension in BF.

In case the wind pressures are supposed to be acting at points C and D (Fig. 18), the condition of affairs remains unchanged so far as moments in the posts are concerned. The only differences with the preceding case are in direct stresses. Since, here

$$S = - \frac{(P_1 + P)(f - h_0)}{b},$$

we get for the direct stresses in posts and braces the following expressions:

$$\overline{AB} = - \frac{(P_1 + P)(f - h_0)}{2e},$$

$$\overline{CD} = - \frac{(P_1 + P)(h - h_0)}{2e} + P_1,$$

$$\overline{AD} = + \frac{(P_1 + P)(f - h_0)}{b \sin \alpha}.$$

It is to be noted that the neglect of direct stresses in the calculation of internal work implies the indeformability of the frame $ABCD$, which is evidently not true, but the effect of its deformation is generally so small that the formulas deduced above will be practically correct.

This form of construction is more common in portal bracing of a metallic bridge than in viaduct bents, of which former, the following example furnishes a case.

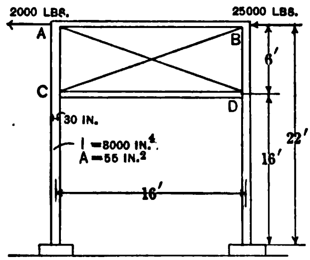

Fig. 19

EXAMPLE. — In the portal bracing of Fig. 19 find the stresses in braces and the greatest fibre stress produced in the posts in carrying the wind pressures as shown, down to the masonry.

From Eq. (33),

$$h_0 = \frac{16\,(44 + 16)}{2\,(22 + 32)} = 8.9 \text{ ft.}$$

From (34) and (35),

$$M_1 = 13{,}500 \times 8.9 = 120{,}150 \text{ ft.-lbs.}$$
$$M_0 = -\,13{,}500\,(16 - 8.9) = -\,95{,}850 \text{ ft.-lbs.}$$

From Eq. (32),

$$S = -\,\frac{27{,}000\,(22 - 8.9)}{16} = -\,22{,}100 \text{ lbs.}$$

Consequently the maximum fibre stress in the post will be

$$\frac{22{,}100}{55} + \frac{120{,}150 \times 12}{8000} \times 15 = 3105 \text{ lbs. per square in.,}$$

being tension in the right post and compression in the left one.

Supposing the diagonal braces to be capable of resisting tension only, only AD would be in action with the given direction of wind pressures, and we get,

$$\overline{AB} = -(2000+25{,}000)\frac{(22-8.9)}{2\times 6} + \frac{2000-25{,}000}{2} = -40{,}980 \text{ lbs.}$$

$$\overline{CD} = -\frac{(2000+25{,}000)(22-8.9)}{2\times 6} = -29{,}480 \text{ lbs.}$$

$$\overline{AD} = \frac{(2000+25{,}000)(22-8.9)}{16\times .35} = +63{,}160 \text{ lbs.}$$

23. In a bent with knee-bracings, such as shown in Fig. 20, assuming all the members to be hinged at their connections and the base of each post firmly anchored to the masonry, we have, as before,

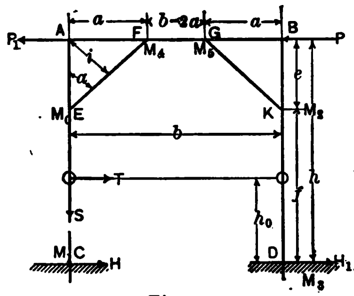

Fig. 20

$$H = H_1 = \frac{P+P_1}{2},$$

$$T = H,$$

$$S = -\frac{(P+P_1)(h-h_0)}{b},$$

$$M_1 = -M_3 = Hh_0,$$
$$M_0 = -M_2 = -H(f-h_0),$$

$$M_4 = -M_5 = -H(h-h_0)\left(\frac{b-2a}{b}\right),$$

h_0 representing the height of the points of contraflexure as in all the previous cases.

Considering bending moments only, we get for ω the following expression; the origins of x being taken at C, E, A, and F:

$$\omega = \frac{1}{EI_2}\int_0^f (M_1 - Hx)^2\,dx + \frac{1}{EI_2}\int_0^e \left(M_0 - \frac{M_0}{e}x\right)^2 dx$$

$$+ \frac{1}{EI_1}\int_0^a \left(\frac{M_4}{a}x\right)^2 dx + \frac{1}{2\,EI_1}\int_0^{b-2a} \left(M_4 - \frac{2\,M_4}{b-2\,a}x\right)^2 dx,$$

in which I_1 and I_2 represent the moments of inertia of the strut AB and posts AC and BD respectively. Since

$$\frac{dM_1}{dh_0} = H,\ \frac{dM_0}{dh_0} = H,\ \frac{dM_4}{dh_0} = H\left(\frac{b-2\,a}{b}\right),$$

we get for

$$\frac{d\omega}{dh_0} = 0,$$

$$\frac{1}{EI_2}\int_0^f 2\,H^2\,(h_0 - x)\,dx + \frac{1}{EI_2}\int_0^e -2\,H^2\,(f - h_0)\left(1 - \frac{x}{e}\right)^2 dx$$

$$+ \frac{1}{EI_1}\int_0^a -2\,H^2\left(\frac{b-2\,a}{ab}\right)^2 (h - h_0)\,x^2\,dx$$

$$+ \frac{1}{2\,EI_1}\int_0^{b-2a} -2\,H^2\,(h - h_0)\left(\frac{b-2\,a}{b}\right)^2\left(1 - \frac{2\,x}{b-2\,a}\right)^2 dx = 0,$$

from which

$$h_0 = \frac{bI_1\,(2\,h + f)\,f + I_2\,(b - 2\,a)^2\,h}{2\,bI_1\,(h + 2f) + I_2\,(b - 2\,a)^2} \quad . \quad . \quad . \ (36)$$

Knowing h_0, the moment at any point of the bent may at once be written. Thus,

C to E, $H\,(h_0 - x)$, origin of x taken at C,

E to A, $-H\,(f - h_0)\left(1 - \frac{x}{e}\right)$, origin of x taken at E,

A to F, $-H\,(h - h_0)\left(\frac{b - 2a}{b}\right)\frac{x}{a}$, origin of x taken at A,

F to G, $-H\,(h - h_0)\left(\frac{b - 2a}{b}\right)\left(1 - \frac{2x}{b - 2a}\right)$, origin of x taken at F.

The moments in the right half of the frame are simply opposite in signs to those of the left. The direct stresses are obtained in the following manner:

Passing a section through A and EF, and considering the left portion above o, the moment taken with respect to A will give:

$$- T\,(h - h_0) - \overline{EF} \cdot i = 0,$$

or

$$\overline{EF} = - \frac{H\,(h - h_0)}{i} = - \overline{GK}.$$

In the same section, since $\Sigma V = 0$,

$$- S + \overline{AE} + \overline{EF} \cos \alpha = 0,$$

from which

$$\overline{AE} = S - \overline{EF} \cos \alpha = \frac{H\,(h - h_0)\,(b - 2a)}{ab} = - \overline{BK},$$

$$\overline{CE} = S = - \frac{2H\,(h - h_0)}{b} = - \overline{KD}.$$

Passing a section through AF and EF, and considering the left position of it, since Σ Horiz. forces $= 0$,

$$H - P_1 + \overline{AF} + \overline{EF} \sin \alpha = 0,$$

from which

$$\overline{AF} = \frac{P_1 - P}{2} + \frac{H\,(h - h_0)}{e}.$$

A vertical section through FG will give for the left half of the section,

$$H - P_1 + \overline{FG} = 0,$$

or

$$\overline{FG} = H + P_1 = \frac{P_1 - P}{2}.$$

At a section through GB and GK we have for the left portion of the same,

$$H - P_1 + \overline{GB} + \overline{GK} \sin \alpha = 0,$$

from which

$$\overline{GB} = \frac{P_1 - P}{2} - \frac{H\,(h - h_0)}{e}.$$

Like the last case, this form of construction is often used for portals, especially of a wooden bridge. The following example is a case of the latter.

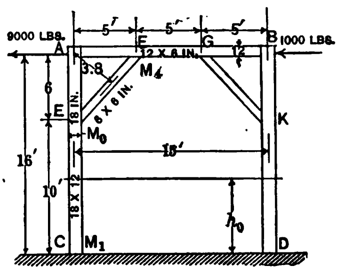

Fig. 21

EXAMPLE. — To calculate the stresses in the portal of a wooden bridge, of Fig. 21, due to wind pressures as shown.

Here $$I_1 = \frac{6 \times 12^3}{12} = 864 \text{ in.}^4,$$

$$I_2 = \frac{12 \times 18^3}{12} = 5832 \text{ in.}^4$$

Substituting in Eq. (36) the values of several terms, we get,

$$h_0 = \frac{15 \times 864 \times 42 \times 10 + 5832 \times 5^2 \times 16}{2 \times 15 \times 864 \times 36 + 5832 \times 5^2} = 7.21 \text{ ft.}$$

Then, since $H = \frac{9000 + 1000}{2} = 5000$ lbs.,

$$M_1 = 5000 \times 7.21 = 36,050 \text{ ft.-lbs.}$$
$$M_0 = -5000\,(10 - 7.21) = -13,950 \text{ ft.-lbs.}$$
$$M_4 = -5000\,(16 - 7.21)\,\frac{5}{15} = -14,650 \text{ ft.-lbs.}$$

The following are direct stresses:

$$\overline{CE} = -\frac{10,000\,(16 - 7.21)}{15} = -5860 \text{ lbs.}$$
$$\overline{EF} = -\frac{5000\,(16 - 7.21)}{3.8} = -11,566 \text{ lbs.}$$
$$\overline{GK} = +11,566 \text{ lbs.}$$
$$\overline{AF} = \frac{9000 - 1000}{2} + \frac{5000\,(16 - 7.21)}{6} = +11,325 \text{ lbs.}$$
$$\overline{FG} = \frac{9000 - 1000}{2} = +4000 \text{ lbs.}$$
$$\overline{GB} = \frac{9000 - 1000}{2} - \frac{5000\,(16 - 7.21)}{6} = -3325 \text{ lbs.}$$

The following comparisons of maximum stresses in members, as calculated on assumptions of fixed and pivoted ends of end posts, show that there is a considerable margin of safety in the statical calculations as generally followed in designing a portal brace of this kind, when the post ends are really fixed.

MAXIMUM STRESSES IN LBS. PER SQ. IN.

End Conditions.	*AC.*			*EF.*	*AB.*		
	Direct.	Bending	Total.	Direct.	Direct.	Bending.	Total.
Fixed . . .	27	668	695	321	157	1221	1378
Hinged . .	49	926	1075	658	240	2223	2463

CHAPTER III

CONTINUOUS GIRDERS

24. LET Fig. 22 represent two consecutive spans of a continuous beam, supposed to be resting on immovable supports of such heights that the girder would be unstrained were it completely unloaded.

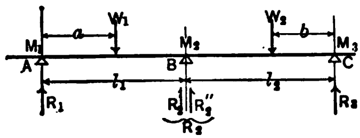

Fig. 22

The following designations will be used throughout the discussion:

M_1, M_2, M_3 . . . moments in the beam at the supports A, B, C respectively.

R_1, R_2' . . . . . reactions at A and B respectively, due to moments and loads in span l_1.

R_2'', R_3 . . . . . reactions at B and C respectively, due to moments and loads in span l_2.

a, b . . . . . . distances of loads from A and C respectively.

I_1, I_2 . . . . . moments of inertia of the beam at AB and BC respectively.

m_1, m_2, m_3, m_4 . . . moments at any points between A and W_1, W_1 and B, B and W_2, W_2 and C respectively.

E . . . . modulus of elasticity assumed to be constant.

Forces acting upward are positive, and vice versâ. Moments causing compression in the upper fibres are positive, and vice versâ.

Tension is taken as positive, and compression, negative. Neglecting the effect of shear, we have for the internal work in the beam,

$$\omega = \frac{1}{2EI_1}\left(\int_0^a m_1^2 dx + \int_a^{l_1} m_2^2 dx\right) + \frac{1}{2EI_2}\left(\int_0^b m_4^2 dx + \int_b^{l_2} m_3^2 dx\right).$$

The reason for taking two consecutive spans as an element of indefinitely continuous girder and confining the summation of internal work to them, lies in the fact that in order to find the value of M_2 which will make ω a minimum, it is unnecessary to go beyond the two spans, since M_2 depends, as will be seen immediately in the following, on M_1 and M_3 and loadings on l_1 and l_2 only. Calling, as before, those moments producing compression in the upper flange positive, and vice versâ, we get the following equations:

$$m_1 = M_1 + R_1 x, \text{ origin of } x \text{ at } A,$$
$$m_2 = M_1 + R_1 x - W_1 (x - a), \text{ origin of } x \text{ at } A,$$
$$m_3 = M_3 + R_3 x - W_2 (x - b), \text{ origin of } x \text{ at } C,$$
$$m_4 = M_3 + R_3 x, \text{ origin of } x \text{ at } C,$$
$$R_1 = \frac{W_1 (l_1 - a)}{l_1} + \frac{M_2 - M_1}{l_1},$$
$$R_3 = \frac{W_2 (l_2 - b)}{l_2} + \frac{M_2 - M_3}{l_2}.$$

Substituting these equations in the expression for work, and setting the first derivative of the same with respect to M_2 equal to zero, we at once obtain the following equation :

$$\frac{d\omega}{dM_2} = \frac{l_1}{I}(M_1 + 2M_2) + \frac{l_2}{I_2}(2M_2 + M_3) + \frac{W_1 a}{l_1 I_1}(l_1^2 - a^2)$$
$$+ \frac{W_2 b}{l_2 I_2}(l_2^2 - b^2) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Whence for any number of loads, we get when $I_1 = I_2$ the following:

$$M_1 l_1 + 2M_2(l_1 + l_2) + M_3 l_2 = -\frac{\sum^{l_1} Wa}{l_1}(l_1^2 - a^2) - \frac{\sum^{l_2} Wb}{l_2}(l_2^2 - b^2). \quad (38)$$

25. For *partial uniform load* w per unit length (Fig. 23), we have but to replace W_1 and W_2 with wda and

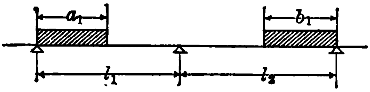

Fig. 23

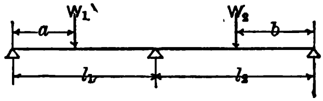

Fig. 24

wdb in (38) and integrate between given limits to obtain the following equation:

$$M_1 l_1 + 2M_2(l_1 + l_2) + M_3 l_2 = -\frac{w}{4l_1}(2l_1^2 a_1^2 - a_1^4) - \frac{w}{4l_2}(2l_2^2 b_1^2 - b_1^4). \, (39)$$

For *full uniform load*, Eq. (39) becomes

$$M_1 l_1 + 2M_2(l_1 + l_2) + M_3 l_2 = -\frac{wl_1^3}{4} - \frac{wl_2^3}{4} \quad . \quad . \quad (40)$$

26. In case both ends of the girder are free and simply supported (Fig. 24), M_1 and M_3 will be equal to zero, so that we get from (38),

$$2M_2(l_1 + l_2) = -\frac{\sum^{l_1} Wa}{l_1}(l_1^2 - a^2) - \frac{\sum^{l_2} Wb}{l_2}(l_2^2 - b^2) \quad . \quad (41)$$

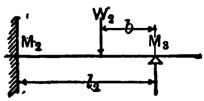

Fig. 25

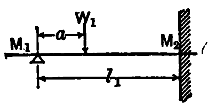

Fig. 26

27. If the left end of the girder were firmly fixed and continuous at the other end (Fig. 25), then it would be equivalent to making $I_1 = \infty$ in Eq. (37), and we get

$$2\,M_2 l_2 + M_3 l_2 = -\frac{\sum^{l_2} Wb}{l_2}(l_2^2 - b^2) \quad . \quad . \quad . \quad (42)$$

28. Similarly if the right end were fixed and continuous at the other (Fig. 26), we would obtain

$$M_1 l_1 + 2\,M_2 l_1 = -\frac{\sum^{l_1} Wa}{l_1}(l_1^2 - a^2) \quad . \quad . \quad . \quad (43)$$

29. When a beam with uniform cross-section is continuous over several supports (Fig. 27), apply Eq. (38)

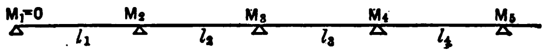

Fig. 27

successively to every two spaces (paying attention to suffixes), in the following manner:

l_1 and l_2, $\quad 0 + 2M_2(l_1 + l_2) + M_3 l_2 = -\dfrac{\sum^{l_1} Wa(l_1^2 - a^2)}{l_1} - \dfrac{\sum^{l_2} Wb(l_2^2 - b^2)}{l_2}.$

l_2 and l_3, $\quad M_2 l_2 + 2M_3(l_2 + l_3) + M_4 l_3 = -\dfrac{\sum^{l_2} Wa(l_2^2 - a^2)}{l_2} - \dfrac{\sum^{l_3} Wb(l_3^2 - b^2)}{l_3}.$

l_3 and l_4, $\quad M_3 l_3 + 2M_4(l_3 + l_4) + M_5 l_4 = -\dfrac{\sum^{l_3} Wa(l_3^2 - a^2)}{l_3} - \dfrac{\sum^{l_4} Wb(l_4^2 - b^2)}{l_4}.$

In this way as many equations as there are unknown moments could be obtained. The rest is a purely algebraic work.

30. In all the foregoing cases of continuous beam, the supports were supposed to be unyielding. If, however, the beam were made to rest on a comparatively yielding support or supports, such as tall metallic columns for instance, then the deformation of the latter would modify the bending moment in the beam by so much as the deflection produced by the sinking of the support makes the beam to take up a portion of the load. Fig. 28 shows a beam continuous over three supports, of which the intermediate one is a column of the same material as the beam.

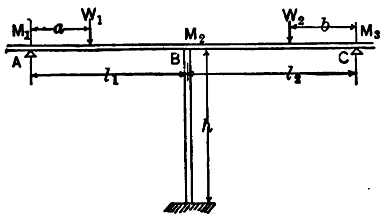

Fig. 28

Using the same designations as before, we have for ω due to W_1 only

$$\omega = \frac{1}{2EI}\left\{\int_0^a m_1^2dx + \int_a^{l_1} m_2^2dx + \int_0^{l_2} m_3^2dx\right\} + \frac{hR_2^2}{2EA},$$

in which A represents the cross-sectional area of the column, and R_2 the pressure acting in the same.

Since

$$R_1 = \frac{W_1(l_1 - a)}{l_1} + \frac{M_2 - M_1}{l_1},$$

$$R_2 = \frac{M_1 - M_2}{l_1} + \frac{M_3 - M_2}{l_2} + \frac{W_1 a}{l_1},$$

$$R_3 = \frac{M_2 - M_3}{l_2},$$

$$R_1 + R_2 + R_3 = W_1,$$

and $m_1 = M_1 + R_1 x$, origin of x at A,
$m_2 = M_1 + R_1 x - W_1 (x - a)$, origin of x at A,
$m_3 = M_3 + R_3 x$, origin of x at C,

substituting these in the equation of work, we get for

$$\frac{dw}{dM_2} = 0,$$

$$\frac{1}{6I}\left\{M_1 l_1 + 2 M_2 (l_1 + l_2) + M_3 l_2 + \frac{W_1 a}{l_1}(l_1^2 - a^2)\right\}$$

$$-\frac{h}{A}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)\left(\frac{M_1 - M_2}{l_1} + \frac{M_3 - M_2}{l_2} + \frac{W_1 a}{l_1}\right) = 0.$$

In case $M_1 = 0$, and $M_3 = 0$, we get

$$M_2 = -\frac{\frac{a}{l_1}\left\{\frac{l_1^2 - a^2}{6I} - \frac{h}{A}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)\right\}}{\frac{l_1 + l_2}{3I} + \frac{h}{A}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)^2} W_1 \quad . \quad . \quad . \quad (44)$$

For W_1 and W_2

$$M_2 = -\frac{\frac{W_1 a}{l_1}\left\{\frac{l_1^2 - a^2}{6I} - \frac{h}{A}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)\right\} + \frac{W_2 b}{l_2}\left\{\frac{l_2^2 - b^2}{6I} - \frac{h}{A}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)\right\}}{\frac{l_1 + l_2}{3I} + \frac{h}{A}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)^2} . (45)$$

If $l_1 = l_2 = l$, we get for W_1

$$M_2 = -\frac{\frac{a(l^2 - a^2)}{6Il} - \frac{2ah}{Al^2}}{\frac{2l}{3I} + \frac{4h}{Al^2}} W_1 \quad . \quad . \quad . \quad (46)$$

whence

$$R_2 = \frac{3al^2 - a^3}{3I\left(\frac{4h}{A} + \frac{2l^3}{3I}\right)} W_1 \quad . \quad . \quad . \quad (47)$$

31. If, owing to any cause, the central support were found, either to yield when loaded or to be so displaced that the beam has to deflect to bear on the supports, the force exerted simply to keep the beam on to the latter produces reactions and moments. That force is no other than R_2 which has for its displacement the deflection of the beam.

Represent by Δh the deflection of the beam at the central support, reckoned in the direction of the force, acting through it, i.e., negative for sinking, and vice versâ. Let M_1, M_2, etc., be moments caused by the motion of the support. Then, according to the first theorem of Castigliano (Art. 6),

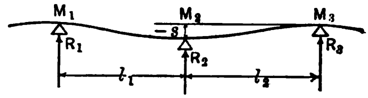

Fig. 29

$$\frac{d\omega}{dR_2} = \Delta h.$$

Since

$$R_2 = \frac{M_1 - M_2}{l_1} + \frac{M_3 - M_2}{l_2},$$

and for the internal work we have as before,

$$\omega = \frac{l_1}{6\,EI}(M_1^2 + M_1M_2 + M_2^2) + \frac{l_2}{6\,EI}(M_2^2 + M_2M_3 + M_3^2),$$

making M_2 the variable,

$$dR_2 = -\left(\frac{1}{l_1} + \frac{1}{l_2}\right)dM_2,$$

we get

$$\frac{d\omega}{dR_2} = -\frac{1}{6\,EI}\left\{l_1(M_1 + 2\,M_2) + l_2(M_3 + 2\,M_2)\right\}\frac{1}{\frac{1}{l_1} + \frac{1}{l_2}} = \Delta h,$$

from which

$$M_1l_1 + 2\,M_2(l_1 + l_2) + M_3l_2 = -6\,EI\Delta h\left(\frac{1}{l_1} + \frac{1}{l_2}\right). \qquad (48)$$

In case the central support sinks by s, then $\Delta h = - s$, and if in that case the ends of the girder were free, we would have

$$2 M_2 (l_1 + l_2) = 6 EIs\left(\frac{1}{l_1} + \frac{1}{l_2}\right) \quad . \quad . \quad . \quad . \quad (49)$$

If, instead of the central support, the left support, for instance, deflect by Δh, then in this case, since

$$\omega = \frac{M_2^{\ 2}}{6 EI} (l_1 + l_2),$$

$$R_1 = \frac{M_2}{l_1},$$

we get

$$\frac{d\omega}{dR_1} = \Delta h = \frac{M_2 l_1 (l_1 + l_2)}{3 EI},$$

whence

$$M_2 = \frac{3 EI\Delta h}{l_1 (l_1 + l_2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Example 1. — A continuous girder with a length of 200 ft., and a uniform section whose depth is 16 ft. and moment of inertia 552,960 in.^{4}, is supported at its centre by a metallic pier 50 ft. high and 50 sq. in. in section. To calculate the maximum stresses found in the chords and pier due to a full uniform load of 3600 lbs. per ft. run.

From Eq. (47),

$$R_2 = 2 \frac{\int_0^l (3 al^2 - a^3) wda}{3 I\left(\frac{4 h}{A} + \frac{2 l^3}{3 I}\right)} = \frac{5 l^4 w}{6 I\left(\frac{4 h}{A} + \frac{2 l^3}{3 I}\right)} = 440{,}000 \text{ lbs.}$$

Since

$$2 R_1 + R_2 - 3600 \times 200 = 0,$$

$$R_1 = 3600 \times 100 - \frac{440{,}000}{2} = 140{,}000 \text{ lbs.}$$

Comparing + and − moments in the girder, the latter will be found to be greater in this case, being at the central support

$$140{,}000 \times 100 - 3600 \times \frac{100^2}{2} = -\ 4{,}000{,}000 \text{ ft.-lbs.},$$

from which we obtain for maximum flange stress in the girder,

$$\frac{4{,}000{,}000 \times 12^2 \times 8}{552{,}960} = 8333 \text{ lbs. per sq. in.},$$

and for the stress in the pier,

$$\frac{440{,}000}{50} = 8800 \text{ lbs. per sq. in.}$$

EXAMPLE 2. — If in the foregoing example the central support were of masonry, so that it might be considered practically indeformable, but owing to yielding foundation, suppose it to settle by .176 in., what would be the moment and reaction at the centre, assuming $E = 30{,}000{,}000$ lbs. per sq. in.?

From Eqs. (41) and (49),

$$M_2 = -\frac{1}{2\,l^2}\int_0^l a\,(l^2 - a^2)\,wda + \frac{3\,EIs}{l^2} = -\frac{wl^2}{8} + \frac{3\,EIs}{l^2}$$

$$= -\frac{3600 \times 100^2}{8} + \frac{3 \times 30{,}000{,}000 \times 552{,}960 \times .176}{100^2 \times 12^3}$$

$$= -\ 3{,}993{,}120 \text{ ft.-lbs.},$$

from which

$$R_2 = 2\left(3600 \times 50 + \frac{3{,}993{,}120}{100}\right) = 440{,}000 \text{ lbs.}$$

32. When the truss forming a continuous girder is of considerable depth, the influence of deformations of web-members which has been neglected in the foregoing discussions becomes felt to some extent. A method of taking the same into consideration will be explained when deducing formulas for swing bridges. Trusses continuous over several supports are, owing to several drawbacks, so seldom constructed, that it will not be necessary to go farther into the subject in this place.

SWING BRIDGE, WITH THREE SUPPORTS

33. For a swing bridge with three supports, when it is of plate girders or trusses of comparatively small depth in which the effect of deformations of web-members is inconsiderable when compared to that of chords, Eq. (41) may be used with correctness sufficient for all practical purposes, and from it other necessary equations may be at once written.

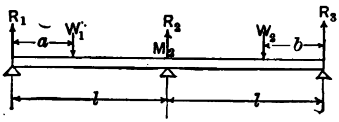

Fig. 30

In (41) making $l_1 = l_2 = l$,

$$M_2 = -\frac{\Sigma W_1 a(l^2 - a^2) + \Sigma W_2 b\,(l^2 - b^2)}{4\,l^2} \quad . \quad . \quad . \quad (51)$$

$$R_1 = \frac{1}{l}\{M_2 + \Sigma W_1 (l - a)\},$$

$$R_2 = \frac{1}{l}(-\,2\,M_2 + \Sigma W_1 a + \Sigma W_2 b),$$

$$R_3 = \frac{1}{l}\{M_2 + \Sigma W_2 (l - b)\},$$

$$R_1 + R_2 + R_3 = \Sigma W_1 + \Sigma W_2.$$

34. When both ends of the bridge are *simply supported* without being raised, the dead-loads act as on two overhanging arms, either when the bridge is closed or open, the live-load alone acting as on a continuous girder on three supports when both arms are loaded partially or fully.

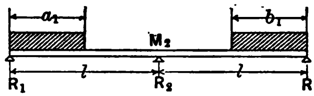

Fig. 31

When the moving load is a uniformly distributed one of w per unit length (Fig. 31), then from Eq. (39),

$$M_2 = -\frac{a_1^2(2l^2 - a_1^2) + b_1^2(2l^2 - b_1^2)}{16l^2}w \quad . \quad . \quad (52)$$

$$R_1 = \frac{M_2}{l} + \frac{wa_1(2l - a_1)}{2l},$$

$$R_2 = -2\frac{M_2}{l} + \frac{w(a_1^2 + b_1^2)}{2l},$$

$$R_3 = \frac{M_2}{l} + \frac{wb_1(2l - b_1)}{2l},$$

$$R_1 + R_2 + R_3 = w(a_1 + b_1).$$

If in this case, one arm only be loaded, then the end of the other arm would be lifted clear of its support, and the loaded arm would be a simple girder with span length l. This mode of loading generally gives maximum positive moment and shear, which are for the left arm at any point distant x from the left end,

$$m = \frac{wx}{2}(l - x) \text{ for full load,}$$

$$s = \frac{w(l - x)^2}{2l} \text{ for load covering } (l - x),$$

s denoting shear taken as positive when it tends to move the left side upward past the right side of the section. A little consideration will show that the greatest negative moment and shear at any point of the left arm will be produced by the greatest negative amount of R_1 combined with load between the point and the left end of the arm. Now from the expression for R_1 it will be seen that all loads on the right arm make R_1 negative, while those on the left arm positive. Consequently the maximum negative moment and shear at any point x (Fig. 32) will be caused by the load covering the right arm and the portion of the left arm between the point and the left end of the arm. They are,

Fig. 32

$$m = R_1 x - \frac{wx^2}{2} = -\left\{ \frac{x^2(2l^2 - x^2) + l^4}{16l^3} - \frac{x(l-x)}{2l} \right\} wx,$$

$$s = R_1 - wx = -\left\{ \frac{x^2(2l^2 - x^2) + l^4}{16l^3} + \frac{x^2}{2l} \right\} w.$$

The absolute maximum negative moment will, for the same reason, be found at the central support when both arms are fully loaded.

These considerations are all that will be necessary in determining maximum stresses in different members of the truss.

35. In case both ends of the bridge are *fully lifted*, the dead-load will be supported on three supports when the bridge is closed, and the central moment due to the same is to be calculated with Eq. (51) or (52).

36. When the girder, instead of being a beam as in the preceding case, is a truss with considerable depth, the deformations of web-members may sometimes be so great that it would be necessary to take them into consideration in accurate calculations. To do this, however, since the dimensions of each member of the truss should be known, it is the general practice to make preliminary calculation of stresses in all the members with the external forces as found by the equations already given for the case of uniform cross-section, with the effect of web-stresses neglected, and afterward to make such tentative corrections as are necessary on the dimensions according to the more accurate computations based on them. The following is an accurate method of determining the external forces.

Let

A = the cross-section of any member of the truss,
E = the modulus of elasticity of the material, assumed to be constant,
S = the stress in the member,
L = the length of the member.

Then for the total internal work in the truss in which the members are subjected to direct stresses only, we have,

$$\omega = \Sigma \frac{S^2 L}{2\,AE} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

In the swing-bridge truss of Fig. 33, since M_2 must always be such as to make the total internal work a minimum, if we now express S in each member due to any given loading in terms of M_2 and substitute it in (53), then from

$$\frac{d\omega}{dM_2} = 0$$

we can at once obtain the required value of M_2. For simplicity, assume two symmetrical loads W, W and distinguish A, S and L of each member with corresponding

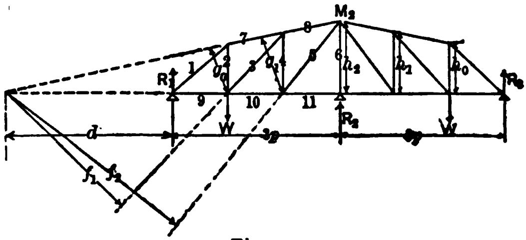

Fig. 33

suffixes and the arm-lengths of several members as shown in the figure. Then taking moments at the successive sections, we get the following values of S;

$$S_1 = - R_1 \frac{L}{h_0},$$

$$S_2 = R_1 \frac{d}{d + p},$$

$$S_3 = \frac{W(d + p) - R_1 d}{f_1},$$

$$S_4 = - \frac{W(d + p) - R_1 d}{d + 2p},$$

$$S_5 = \frac{W(d + p) - R_1 d}{f_2},$$

$$S_6 = - R_2,$$

$$S_7 = - \frac{R_1 p}{g_0},$$

$$S_8 = - \frac{(2 R_1 - W) p}{g_1},$$

$$S_9 = R_1 \frac{p}{h_0},$$

$$S_{10} = \frac{(2 R_1 - W) p}{h_1},$$

$$S_{11} = \frac{(3 R_1 - 2 W) p}{h_2};$$

and since

$$R_1 = \frac{M_2}{3 p} + \frac{2 W}{3},$$

$$R_2 = - \frac{2 M_2}{3 p} + \frac{2 W}{3},$$

$$R_3 = \frac{M_2}{3 p} + \frac{2 W}{3},$$

$$R_1 + R_2 + R_3 = 2 W.$$

Substituting, we get,

$$\omega = \frac{1}{2E} \left\{ 2 R_1^2 \frac{L_1^3}{h_0^2 A_1} + 2 R_1^2 \frac{d^2 h_0}{(d+p)^2 A_2} + \cdots + R_2^2 \frac{h_2}{A_6} + 2 R_1^2 \frac{p^2 L_7}{g_0^2 A_7} + \cdots \right\}.$$

Setting the first derivative of ω with respect to M_2 equal to zero, remembering that,

$$\frac{dR_1}{dM_2} = \frac{dR_3}{dM_2} = \frac{1}{3 p},$$

$$\frac{dR_2}{dM_2} = - \frac{2}{3 p},$$

we get,

$$R_1 = \frac{d(d+p) \left\{ \frac{L_3}{f_1^2 A_3} + \frac{L_4}{(d+2p)^2 A_4} + \frac{L_5}{f_2^2 A_5} \right\} + \frac{2 h_2}{A_6} + p^2 \left(\frac{2 L_8}{g_1^2 A_8} + \frac{2p}{h_1^2 A_{10}} + \frac{6p}{h_2^2 A_{11}} \right)}{\frac{L_1^3}{h_0^2 A_1} + d^2 \left\{ \frac{h_0}{(d+p)^2 A_2} + \frac{L_3}{f_1^2 A_3} + \frac{L_4}{(d+2p)^2 A_4} + \frac{L_5}{f_2^2 A_5} \right\} + \frac{2 h_2}{A_6} + p^2 \left(\frac{L_7}{g_0^2 A_7} + \frac{4 L_8}{g_1^2 A_8} + \frac{p}{h_0^2 A_9} + \frac{4p}{h_1^2 A_{10}} + \frac{9p}{h_2^2 A_{11}} \right)}$$

whence M_2 and R_2 may be obtained by substitution in the foregoing equations.

In this way R_1, R_2, and M_2 are to be calculated for different modes of loading in order to obtain stresses due to them.

A comparison of approximate and correct methods of computation shows that the difference in results obtained by the two methods is generally inconsiderable, as will be shown in the case of a swing bridge with four supports (Art. 37).

SWING BRIDGE, WITH FOUR SUPPORTS AND PARTIALLY CONTINUOUS

37. A swing bridge *fully continuous* over four supports has probably never been constructed, owing to practical difficulties in construction arising from the great difference in amounts between central reactions when the bridge is partially loaded, which may necessitate special provisions for holding down the central supports on to the masonry. For this reason, such bridge is made either partially continuous or entirely discontinuous. Partial continuity in such case is effected by omitting the web-members between central supports, thus cutting off the means of transmitting shear from one span to the other. Fig. 34 shows this kind of construction. As there can be no shear in the panel BC, it is evident that M_2 will always be equal to M_2'.

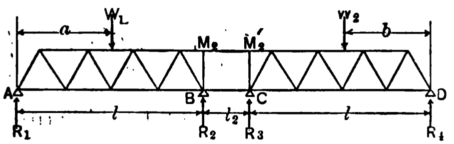

Fig. 34

Represent by

m_1 = moment at any point x from A between A and W_1.
m_2 = moment at any point x from A between W_1 and B.
m_3 = moment at any point x from D between C and W_2.
m_4 = moment at any point x from D between W_2 and D.

Assuming the truss to have uniform moment of inertia I, and neglecting the influence of deformations of web-members, we get for the internal work due to moments:

$$\omega = \frac{1}{2EI}\left\{\int_0^a m_1^2 dx + \int_a^l m_2^2 dx + \int_0^{l_2} M_2^2 dx + \int_b^l m_3^2 dx + \int_0^b m_4^2 dx\right\}..$$

But

$$m_1 = R_1 x = \frac{1}{l}\{M_2 + W_1(l-a)\}x,$$

$$m_2 = R_1 x - W_1(x-a) = \frac{M_2}{l}x + \frac{W_1 a(l-x)}{l},$$

$$m_3 = R_4 x - W_2(x-b) = \frac{M_2}{l}x + \frac{W_2 b(l-x)}{l},$$

$$m_4 = R_4 x = \frac{1}{l}\{M_2 + W_2(l-b)\}x.$$

Substituting these values of m in the above expression for work, we get for

$$\frac{d\omega}{dM_2} = 0,$$

the following equation:

$$2\{M_2 + W_1(l-a)\}\frac{l}{3} + 2M_2 l_2 + 2\{M_2 + W_2(l-b)\}\frac{l}{3}$$

$$-W_1\left(\frac{2l^2}{3} - al + \frac{a^3}{3l}\right) - \left(\frac{2l^2}{3} - bl + \frac{b^3}{3l}\right)W_2 = 0,$$

from which generally for any number of W_1 and W_2 we get

$$M_2 = -\frac{\Sigma W_1 a\,(l^2 - a^2)}{l\,(6l_2 + 4\,l)} - \frac{\Sigma W_2 b\,(l^2 - b^2)}{l\,(6\,l_2 + 4\,l)} \quad . \quad . \quad (54)$$

Knowing M_2, all the external forces become at once known, thus:

$$R_1 = \{M_2 + \Sigma W_1 (l - a)\}\,\frac{1}{l},$$

$$R_2 = (-M_2 + \Sigma W_1 a)\,\frac{1}{l},$$

$$R_3 = (-M_2 + \Sigma W_2 b)\,\frac{1}{l},$$

$$R_4 = \{M_2 + \Sigma W_2 (l - b)\}\,\frac{1}{l},$$

$$R_1 + R_2 + R_3 + R_4 = \Sigma W_1 + \Sigma W_2.$$

These equations give approximate results for most kinds of trusses; a more accurate result is obtained by taking the deformations of the web-members into consideration, and forming

$$\omega = \frac{\Sigma S^2 L}{2\,EA},$$

extended over all the members of the truss, as explained in the case of three supports, the necessary A and S being provisionally obtained by means of the approximate equations above given. The mode of loading to give maximum moment or shear at any point in the truss is essentially the same as in the case of three supports.

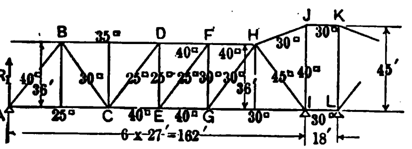

Fig. 35

EXAMPLE. — In the swing bridge of Fig. 35 to calculate the reactions due to uniform moving load of 10 tons per panel, when both arms are fully loaded.

For the assumed position of moving load,

$$R_2 = 50 - R_1.$$

The following stresses and internal works may now be written :

Members.		S.	L (ft.).	A (sq. in.).	$\frac{S^2L}{2EA}$.
Web-members.	AB	$1.25\ R_1$	45	40	$1.76\ R_1^2\left(\frac{1}{E}\right)$
	BC	$1.25\ (R_1 - 10)$	45	30	$2.34\ (R_1 - 10)^2$ (")
	CD	$1.25\ (R_1 - 20)$	45	25	$2.81\ (R_1 - 20)^2$ (")
	DE	$R_1 - 20$	36	25	$1.44\ (R_1 - 20)^2$ (")
	EF	$1.25\ (R_1 - 30)$	45	25	$2.81\ (R_1 - 30)^2$ (")
	FG	$R_1 - 30$	36	30	$1.2\ (R_1 - 30)^2$ (")
	GH	$1.25\ (R_1 - 40)$	45	30	$2.34\ (R_1 - 40)^2$ (")
	HI	$2.2\ (R_1 - 36.4)$	45	45	$4.84\ (R_1 - 36.4)^2$ (")
	JI	$1.2\ (R_1 - 25)$	45	40	$1.62\ (R_1 - 25)^2$ (")
Up. Chd.	BD	$1.5\ (R_1 - 5)$	54	35	$3.47\ (R_1 - 5)^2$ (")
	DF	$2.25\ (R_1 - 10)$	27	40	$3.42\ (R_1 - 10)^2$ (")
	FH	$3\ (R_1 - 15)$	27	40	$6.08\ (R_1 - 15)^2$ (")
	HJ	$3.79\ (R_1 - 25)$	28.5	30	$13.68\ (R_1 - 25)^2$ (")
	JK	$3.6\ (R_1 - 25)$	18	30	$7.78\ (R_1 - 25)^2\left(\frac{1}{2E}\right)$
Low. Chd.	AC	$0.75\ R_1$	54	25	$1.21\ R_1^2\ \left(\frac{1}{E}\right)$
	CE	$2.25\ (R_1 - 10)$	27	40	$3.42\ (R_1 - 10)^2$ (")
	EG	$3\ (R_1 - 15)$	27	40	$6.08\ (R_1 - 15)^2$ (")
	GI	$3.75\ (R_1 - 20)$	54	30	$25.31\ (R_1 - 20)^2$ (")
	IL	$3.6\ (R_1 - 25)$	18	30	$7.78\ (R_1 - 25)^2\left(\frac{1}{2E}\right)$

Summing up the works, and putting the first derivative of the sum with respect to R_1 equal to zero, we at once get,

$$R_1 = 20.19 \text{ tons},$$
$$R_2 = 50 - R_1 = 29.81 \text{ tons}.$$

38. Had the moment of inertia been assumed to be uniform throughout the girder and at the same time the deformation of web-members neglected, we would have obtained from Eq. (54),

$$M_2 = -2\frac{\Sigma\, 10\, a\,(26{,}244 - a^2)}{162\,(108 + 648)}.$$

Substituting in this, $a = 27$, 54, 81, 108, and 135, we obtain

$$M_2 = -1012.5 \text{ ft.-tons,}$$

so that

$$R_1 = \{M_2 + \Sigma 10\,(162 - a)\}\,\tfrac{1}{162} = 18.75 \text{ tons.}$$

Comparing this with the preceding result, it will be seen that the assumption of uniform moment of inertia and the neglect of web-member deformations give R_1 smaller by about 7 per cent in this case than given by the more correct calculation. In practice, however, all this nicety in calculation becomes almost valueless, owing to the overwhelming disturbance brought about by unequal temperature changes, which constantly tend to throw out of adjustment the end supports on which the stresses of all the members solely depend.

DOUBLE-SWING BRIDGE

39. Double-swing bridges are latched at the centre when closed, thus transmitting shear, but no moment from one span to another.

Fig. 36 shows a double-swing bridge with four supports. The point C serves for both trusses as a common yielding support.

To simplify the discussion, all the spans will be made alike. Then for any load W_1 we get the following moments in the several spans:

R_1x	between A and W_1,	distant x from A.
$R_1x - W_1(x - a)$.	between W_1 and B,	distant x from A.
$M_1 + R_2''x$. . .	between B and C,	distant x from B.
$M_2 + R_3'x$ . . .	between D and C,	distant x from D.
R_4x	between E and D,	distant x from E.

Assuming the cross-section of the trusses to be uni-

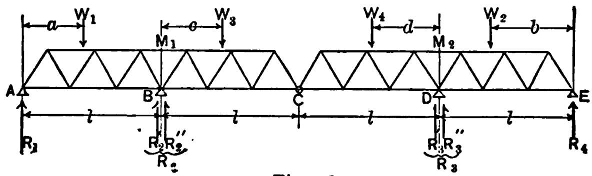

Fig. 36

form throughout, and considering moments only, we get for the total internal work due to W_1:

$$\omega = \frac{1}{2EI}\left[\int_0^a (R_1x)^2\,dx + \int_a^l \{R_1x - W_1(x-a)\}^2\,dx + \int_0^l (M_1 + R_2''x)^2\,dx + \int_0^l (M_2 + R_3'x)^2\,dx + \int_0^l (R_4x)^2\,dx\right].$$

Taking moments successively at B, A, C, E, and D. we get,

$$R_1 = \frac{W_1(l-a)}{l} + \frac{M_1}{l},$$

$$R_2' = \frac{W_1a}{l} - \frac{M_1}{l},$$

$$R_2'' = -\frac{M_1}{l},$$

$$R_2 = R_2' + R_2'' = -\frac{2M_1}{l} + \frac{W_1a}{l},$$

$$R_3' = -\frac{M_2}{l},$$

$$R_3'' = -\frac{M_2}{l},$$

$$R_4 = \frac{M_2}{l}.$$

Taking moment at D,

$$3\,R_1 l - W_1\,(3\,l - a) + 2\,R_2 l = M_2.$$

Substituting the values of R_1 and R_2 in this, we get

$$M_2 = -\,M_1.$$

Introducing these equations in the expression for the internal work, and setting the first derivative of it with respect to M_1 equal to zero, we get

$$M_1 = -\,\frac{a\,(l^2 - a^2)}{8\,l^2}\,W_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

and consequently,

$$M_2 = \frac{a\,(l^2 - a^2)}{8\,l^2}\,W_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Similarly for load W_2 in the right-end span we get,

$$M_1 = \frac{b\,(l^2 - b^2)}{8\,l^2}\,W_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

$$M_2 = -\,\frac{b\,(l^2 - b^2)}{8\,l^2}\,W_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

For any load W_3 in the second span from the left, we have as before the following moments:

$R_1 x$ . . . . between A and B, origin of x at A,
$M_1 + R_2'' x$ . . . . . between B and W_3, origin of x at B,
$M_1 + R_2'' x - W_3\,(x - c)$, between W_3 and C, origin of x at B,
$M_2 + R_3' x$ . . . . . between D and C, origin of x at D,
$R_4 x$ . . . . between E and D, origin of x at E,

from which we get the following internal work:

$$\omega = \frac{1}{2EI}\left[\int_0^l (R_1x)^2\,dx + \int_0^c (M_1 + R_2''x)^2\,dx\right.$$

$$+ \int_c^l \{M_1 + R_2''x - W_3(x-c)\}^2 dx + \int_0^l (M_2 + R_3'x)^2\,dx$$

$$\left. + \int_0^l (R_4x)^2\,dx\right].$$

Taking moments at B, C, and D successively, we have:

$$R_1 = \frac{M_1}{l},$$

$$R_2'' = -\frac{M_1}{l} + \frac{W_3(l-c)}{l},$$

$$R_3' = -\frac{M_2}{l},$$

$$R_4 = \frac{M_2}{l},$$

$$M_2 = -(M_1 + W_3C).$$

Substituting these values in the above expression for work, and putting as before the first differential coefficient of ω with respect to M_1 equal to zero, we at once obtain:

$$M_1 = -\frac{c(6l^2 - 3lc + c^2)}{8l^2}W_3 \quad . \quad . \quad . \quad . \quad (59)$$

$$M_2 = -\frac{c(2l^2 + 3lc - c^2)}{8l^2}W_3 \quad . \quad . \quad . \quad . \quad (60)$$

Similarly for any load W_4 in the third span from the left, we get,

$$M_1 = -\frac{d(2l^2 + 3ld - d^2)}{8l^2}W_4 \quad . \quad . \quad . \quad . \quad (61)$$

$$M_2 = -\frac{d(6l^2 - 3ld + d^2)}{8l^2}W_4 \quad . \quad . \quad . \quad . \quad (62)$$

Finally, we get for *any number of loads,*

$$M_1 = \frac{1}{8l^2}\{-\Sigma W_1 a(l^2 - a^2) + \Sigma W_2 b(l^2 - b^2) - \Sigma W_3 c(6l^2 - 3lc + c^2)$$
$$- \Sigma W_4 d(2l^2 + 3ld - d^2)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

$$M_2 = \frac{1}{8l^2}\{\Sigma W_1 a(l^2 - a^2) - \Sigma W_2 b(l^2 - b^2) - \Sigma W_3 c(2l^2 + 3lc - c^2)$$
$$- \Sigma W_4 d(6l^2 - 3ld + d^2)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

$$R_1 = \frac{\Sigma W_1(l - a)}{l} + \frac{M_2}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

$$R_2 = \frac{\Sigma W_1 a}{l} - \frac{2M_1}{l} + \frac{\Sigma W_3(l - c)}{l} \quad . \quad . \quad . \quad . \quad (66)$$

$$R_3 = \frac{\Sigma W_2 b}{l} - \frac{2M_2}{l} + \frac{\Sigma W_4(l - d)}{l} \quad . \quad . \quad . \quad . \quad (67)$$

$$R_4 = \frac{\Sigma W_2(l - b)}{l} + \frac{M_2}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

$$R_1 + R_2 + R_3 + R_4 = \Sigma W.$$

It has been assumed in the foregoing discussions that the ends A and E of the trusses are not lifted off the supports under all conditions of loading.

40. Fig. 37 shows a double-swing bridge with six sup-

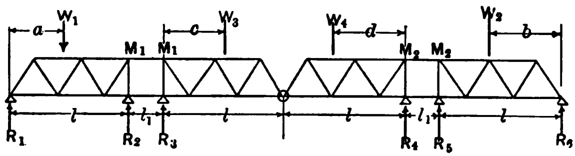

Fig. 37

ports, made partially continuous by the omission of diagonals in the central panel of each truss, for reasons already stated in the case of a common swing bridge.

Then, since by construction there can be no shear in the central panels, the moments over supports belonging to them must be equal to each other in both spans.

Assuming the cross-section of the trusses to be uniform throughout, and considering moments only, we have for total internal work due to any one load W_1 in the left-end span:

$$\omega = \frac{1}{2EI}\Big[\int_0^a (R_1x)^2\,dx + \int_a^l \{R_1x - W_1(x-a)\}^2 dx + \int_0^{l_1} M_1^2 dx$$
$$+ \int_0^l (M_1 + R_3x)^2\,dx + \int_0^l (M_2 + R_4x)^2 dx + \int_0^{l_1} M_2^2 dx$$
$$+ \int_0^l (R_6x)^2\,dx\Big].$$

Since

$$R_1 = \frac{W_1(l-a)}{l} + \frac{M_1}{l},$$
$$R_3 = -\frac{M_1}{l},$$
$$R_4 = -\frac{M_2}{l},$$
$$R_6 = \frac{M_2}{l},$$
$$M_2 = M_1 + 2R_3l = -M_1.$$

Substituting these values in the expression for work, and setting the first derivative of the latter with respect to M_1 equal to zero, we get

$$M_1 = -\frac{a(l^2 - a^2)}{l(8l + 12l_1)}W_1 \quad . \quad . \quad . \quad . \quad . \quad (69)$$

whence, also,

$$M_2 = \frac{a(l^2 - a^2)}{l(8l + 12l_1)}W_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

Similarly for any load W_2 on the right-end span, we get,

$$M_1 = \frac{b(l^2 - b^2)}{l(8l + 12l_1)}W_2 \quad . \quad . \quad . \quad . \quad . \quad (71)$$

$$M_2 = -\frac{b(l^2 - b^2)}{l(8l + 12l_1)}W_2 \quad . \quad . \quad . \quad . \quad (72)$$

For any one load W_3 in the second span from the left, the internal work due to moments caused by the same may be expressed as follows:

$$\omega = \frac{1}{2EI}\left[\int_0^l (R_1 x)^2\,dx + \int_0^{l_1} M_1^2\,dx + \int_0^c (M_1 + R_3 x)^2\,dx \right.$$
$$+ \int_c^l \{M_1 + R_3 x - W_3(x-c)\}^2\,dx$$
$$\left. + \int_0^l (M_2 + R_4 x)^2\,dx + \int_0^{l_1} M_2^2\,dx + \int_0^l (R_6 x)^2\,dx\right].$$

Since in this case,

$$R_1 = \frac{M_1}{l},$$
$$R_3 = \frac{W_3(l-c)}{l} - \frac{M_1}{l},$$
$$R_4 = -\frac{M_2}{l},$$
$$R_6 = \frac{M_2}{l},$$
$$M_2 = -(M_1 + W_3 c).$$

Substituting them in the expression for internal work, and setting the differential coefficient with respect to M_1 equal to zero, we get,

$$M_1 = -\frac{6cl(l+l_1) - c^2(3l-c)}{l(8l+12l_1)}W_3 \quad . \quad . \quad . \quad (73)$$

$$M_2 = -\frac{2cl(l+3l_1) + c^2(3l-c)}{l(8l+12l_1)}W_3 \quad . \quad . \quad (74)$$

Similarly for any load W_4 in the third span from the left, we get,

$$M_1 = -\frac{2dl(l+3l_1) + d^2(3l-d)}{l(8l+12l_1)}W_4 \quad . \quad . \quad (75)$$

$$M_2 = -\frac{6dl(l+l_1) - d^2(3l-d)}{l(8l+12l_1)}W_4 \quad . \quad . \quad (76)$$

Finally, for any number of loads we get,

$$M_1 = \frac{1}{l(8l + 12l_1)}[-\Sigma W_1 a(l^2 - a^2) + \Sigma W_2 b(l^2 - b^2) - \Sigma W_3 c\{6l(l + l_1) - c(3l - c)\} - \Sigma W_4 d\{2l(l + 3l_1) + d(3l - d)\}] \quad (77)$$

$$M_2 = \frac{1}{l(8l + 12l_1)}[\Sigma W_1 a(l^2 - a^2) - \Sigma W_2 b(l^2 - b^2) - \Sigma W_3 c\{2l(l + 3l_1) + c(3l - c)\} - \Sigma W_4 d\{6l(l + l_1) - d(3l - d)\}] \quad (78)$$

$$R_1 = \frac{\Sigma W_1(l - a)}{l} + \frac{M_1}{l} \quad (79)$$

$$R_2 = \frac{\Sigma W_1 a}{l} - \frac{M_1}{l} \quad (80)$$

$$R_3 = \frac{\Sigma W_3(l - c)}{l} - \frac{M_1}{l} \quad (81)$$

$$R_4 = \frac{\Sigma W_4(l - d)}{l} - \frac{M_2}{l} \quad (82)$$

$$R_5 = \frac{\Sigma W_2 b}{l} - \frac{M_2}{l} \quad (83)$$

$$R_6 = \frac{\Sigma W_2(l - b)}{l} + \frac{M_2}{l} \quad (84)$$

$$\Sigma R = \Sigma W.$$

41. The foregoing equations for double-swing bridge give but approximate results for reasons already explained. To obtain more correct results, resort must be had to

$$\omega = \frac{\Sigma S_2 L}{2EA}$$

for expressing the internal work, extending the expression over all the members of the truss, based on the approximate values of S and A provisionally found by the foregoing equations, exactly as explained in the case of common swing bridges.

CHAPTER IV

ARCHES, WITH TWO HINGES

42. In an arch with hinges at both ends, since the moments cannot exist at these points, the reactions ought to pass through the latter.

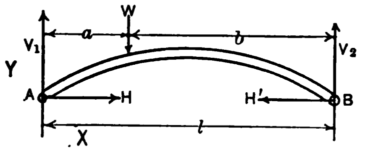

Fig. 38

Fig. 38 shows a symmetrical arch-rib with hinges at A and B.

The following designations will be used throughout the discussion:

V_1 and H	the vertical and horizontal components of the reaction at A.
V_2 and H'	ditto at B.
l	span length.
l'	length of arch measured along the axis of the rib.
x and y	coördinates with origin at A.
c	distance from A measured along the axis of the rib.
ϕ	inclination of tangent at x, y to the horizontal.
a and b	distances of a load from A and B respectively.
a'	the distance measured along the axis of the rib from A.
m	moment at any point.
N	normal stress at a section.
T	tangential stress at the section.
R	resultant force.
E	modulus of elasticity assumed to be constant.

I . . . . . moment of inertia of the normal section of the rib.
A . . . . . cross-sectional area of the rib.

Forces acting upward are taken as positive.

Moments producing compression at the extrados are positive.

Forces acting toward right are positive.

Tensions are positive and all vice versâ.

Since equilibrium requires that among external forces as well as between external and internal forces

Σ horiz. forces = o, Σ vert. forces = o, and Σ moments = o,

we have,

$$H - H' = 0,$$
$$V_1 + V_2 - W = 0;$$

and from moments taken with respect to B and A,

$$V_1 = \frac{b}{l} W,$$

$$V_2 = \frac{a}{l} W.$$

43. Since at any section of the rib, wherever the resultant of external forces does not pass through the centre of gravity, a moment will be caused at the section, and further, if the direction of the resultant does not coincide with that of the tangent to the axis of the rib, the latter, beside being axially compressed, will be subjected to tangential stress at the section.

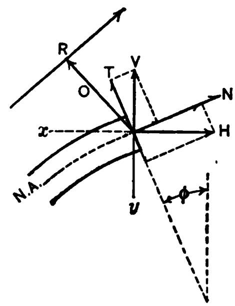

Fig. 39

At any point x, y of the neutral axis of the rib, then, referring to Figs. 38 and 39, we have,

$$m = Ro = V_1 x - Hy, \text{ for } x < a$$
$$= V_1 x - W(x - a) - Hy, \text{ for } x > a.$$

Decomposing R into V and H we get,

$$N + V \sin \phi + H \cos \phi = 0,$$
$$T + V \cos \phi - H \sin \phi = 0,$$

in which

$$V = \frac{b}{l} W \text{ for } x < a,$$
$$= \left(\frac{b}{l} - 1\right) W \text{ for } x > a;$$

and since the loading is vertical, H will be constant throughout the arch.

44. Neglecting the effect of tangential stress for the reason already stated (Art. 5), we have for the internal work in the rib due to W:

$$\omega = \int_0^{l'} \frac{m^2 dc}{2IE} + \int_0^{l'} \frac{N^2 dc}{2AE},$$

in which

$$dc = \frac{dx}{\cos \phi},$$

being the elementary length measured along the axis of the rib.

Substituting in this expression for work, the values of m and N already given, we get:

$$\omega = \int_0^{a'} \frac{(V_1 x - Hy)^2 dc}{2IE} + \int_{a'}^{l'} \frac{\{V_1 x - Hy - W(x - a)\}^2 dc}{2IE}$$
$$+ \int_0^{a'} \frac{(V \sin \phi + H \cos \phi)^2 dc}{2AE} + \int_{a'}^{l'} \frac{(V \sin \phi + H \cos \phi)^2 dc}{2AE}.$$

Since H must make ω a minimum, we obtain for

$$\frac{d\omega}{dH} = 0$$

the following:

$$\int_0^{a'} - \frac{Wbxy}{lI} dc + \int_0^{a'} \frac{Hy^2}{I} dc + \int_{a'}^{l'} - \frac{Wbxy}{lI} dc + \int_{a'}^{l'} \frac{W(x-a)y}{I} dc$$
$$+ \int_{a'}^{l'} \frac{Hy^2}{I} dc + \int_0^{a} \frac{Wb \sin \phi}{lA} dx + \int_0^{a} \frac{H \cos \phi}{A} dx$$
$$+ \int_a^{l} \frac{W(b-l) \sin \phi}{lA} dx + \int_a^{l} \frac{H \cos \phi}{A} dx = 0,$$

from which,

$$H = \frac{\int_0^{l'} \frac{bxy}{lI} dc - \int_{a'}^{l'} \frac{(x-a)y}{I} dc - \int_0^{l} \frac{b \sin \phi}{lA} dx + \int_a^{l} \frac{\sin \phi}{A} dx}{\int_0^{l'} \frac{y^2}{I} dc + \int_0^{l} \frac{\cos \phi}{A} dx} W \qquad (85)$$

45. This equation could be somewhat simplified by taking, instead of one one-sided loading, two symmetrical loads, which will evidently give H simply double that for single one. Referring to Fig. 40, and extending the integral over one-half the arch, we get the following expression for H due to one W:

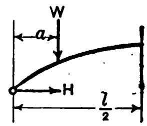

Fig. 40

$$H = \frac{1}{2} \frac{\int_0^{a'} \frac{xydc}{I} + a \int_{a'}^{\frac{l'}{2}} \frac{ydc}{I} - \int_0^{a} \frac{\sin \phi \, dx}{A}}{\int_0^{\frac{l'}{2}} \frac{y^2dc}{I} + \int_0^{\frac{l}{2}} \frac{\cos \phi \, dx}{A}} W \quad . \quad . \quad (86)$$

Approximate results may be obtained by neglecting the effect of the normal or *axial stress*, which is generally inconsiderable when compared to that of the moment. For this it is simply necessary to leave out the terms containing N and A in the preceding equations, so that we get from Eq. (85),

$$H = \frac{\int_0^{l'} \frac{bxy}{lI} dc - \int_{a'}^{l'} \frac{(x-a)\,y}{I} dc}{\int_0^{l'} \frac{y^2 dc}{I}} W \quad . \quad . \quad . \quad (87)$$

or from Eq. (86),

$$H = \frac{1}{2} \frac{\int_0^{a'} \frac{xydc}{I} + a \int_{a'}^{\frac{l'}{2}} \frac{ydc}{I}}{\int_0^{\frac{l'}{2}} \frac{y^2 dc}{I}} W \quad . \quad . \quad . \quad . \quad (88)$$

46. Temperature Stresses.—A temperature change causes variation in the length of the rib, and were the arch-end free to move, a corresponding change would take place in span length; but as the supports are here assumed to be immovable, the rib is forced back, as it were, to its supports.

Let

t = temperature change in number of degrees,
θ = coefficient of expansion and contraction,
H_t = horizontal reaction at the left support due to the temperature change.

Imagine the arch to be free to move at the left end, then the force H_t exerted at the support must be sufficient to force the arch through a distance of

$$t\theta l,$$

reckoned in the direction of the force, i.e., *positive* for the *rising* temperature. Then, according to the first theorem of Castigliano,

$$\frac{d\omega}{dH_t} = t\theta l.$$

Using the same designations as before,

$$\omega = \int_0^{l'} \frac{m^2 dc}{2\,IE} + \int_0^{l'} \frac{N^2 dc}{2\,AE},$$

m and N here representing moment and normal stress due to H_t.

Since

$$m = -H_t y,$$
$$N = -H_t \cos\phi,$$
$$\omega = H_t^2\left(\int_0^{l'} \frac{y^2 dc}{2\,IE} + \int_0^{l'} \frac{\cos^2\phi\, dc}{2\,AE}\right),$$

whence

$$\frac{d\omega}{dH_t} = H_t\left(\int_0^{l'} \frac{y^2 dc}{IE} + \int_0^{l} \frac{\cos\phi\, dx}{AE}\right) = t\theta l,$$

from which

$$H_t = \frac{t\theta l E}{\int_0^{l'} \frac{y^2 dc}{I} + \int_0^{l} \frac{\cos\phi\, dx}{A}} \quad . \quad . \quad . \quad . \quad . \quad (89)$$

Neglecting the effect of axial stress, we get,

$$H_t = \frac{t\theta l E}{\int_0^{l'} \frac{y^2 dc}{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

Eqs. (85) to (90) will give the amount of horizontal reaction for vertical loadings and *uniform* changes of temperature when the form of the arch is known.

47. Displacement of Supports. — If, owing to yielding or settling of supports, a change in span length takes place, the effect on the arch would be similar to that due to temperature changes.

Let

Δl = change in span length measured at the left support in the direction of the force causing the same, i.e., ***negative*** for the ***increase of span length***, and vice versâ.

H_{Δ} = horizontal reaction at the left support due to change in span length.

Then, since

$$\omega = H_{\Delta}^2 \left(\int_0^{l'} \frac{y^2 dc}{2\,IE} + \int_0^{l} \frac{\cos\phi\, dx}{2\,AE} \right),$$

by the same reasoning as before, we have

$$\frac{d\omega}{dH_{\Delta}} = H_{\Delta} \left(\int_0^{l'} \frac{y^2 dc}{IE} + \int_0^{l} \frac{\cos\phi\, dx}{AE} \right) = \Delta l,$$

from which

$$H_{\Delta} = \frac{E\Delta l}{\displaystyle\int_0^{l'} \frac{y^2 dc}{I} + \int_0^{l} \frac{\cos\phi\, dx}{A}}. \quad . \quad . \quad . \quad . \quad . \quad (91)$$

Neglecting the effect of axial stress, we get,

$$H_{\Delta} = \frac{E\Delta l}{\displaystyle\int_0^{l'} \frac{y^2 dc}{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

The effect of slight *changes in the heights of supports* is generally so small in this kind of arches, that it is unnecessary to take them into consideration in the calculation of stress due to displacement of supports.

PARABOLIC ARCH WITH TWO HINGES

48. If we assume the cross-section of the rib to increase from the crown toward both ends in such a way that at any point

$$I = I_0 \sec \phi,$$
$$A = A_0 \sec \phi,$$

in which I_0 and A_0 denote the moment of inertia and the cross-sectional area of the rib at the crown, the calculation of stresses in a parabolic arch-rib becomes considerably simplified. Thus, introducing in Eq. (86) the equation of parabola with origin at A (Fig. 41),

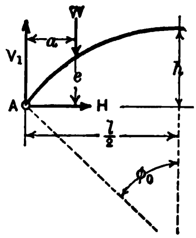

Fig. 41

$$y = \frac{4h}{l^2} x (l - x),$$

and remembering that

$$dx = \cos \phi \, dc,$$
$$\sin \phi \, dx = \cos \phi \, dy,$$

we get,

$$H = \frac{1}{2} \frac{\frac{4h}{l^2 I_0} \left\{ \int_0^a x^2 (l-x) dx + a \int_a^{\frac{l}{2}} x (l-x) \, dx \right\} - \frac{1}{A_0} \int_0^a \cos^2 \phi \, dy}{\frac{16 h^2}{l^4 I_0} \int_0^{\frac{l}{2}} x^2 (l - x)^2 \, dx + \frac{1}{A_0} \int_0^{\frac{l}{2}} \cos^2 \phi \, dx} W.$$

Since

$$\int_0^a x^2 (l - x) \, dx + a \int_a^{\frac{l}{2}} x (l - x) \, dx = \frac{a}{12} (a^3 - 2 a^2 l + l^3),$$

$$\int_0^a \cos^2 \phi \, dy = \frac{l^2 e}{l^2 + 16 h^2} \text{ nearly,*}$$

$$\int_0^{\frac{l}{2}} x^2 (l - x)^2 \, dx = \frac{l^5}{60},$$

$$\int_0^{\frac{l}{2}} \cos^2 \phi \, dx = \frac{l^2}{8h} \phi_0,$$

* Howe, Treatise on Arches.

e denoting the value of y at a; and ϕ_0, the inclination of the tangent at A to the horizontal.

Putting

$$\frac{I_0}{A_0} = i^2,$$

we get,

$$H = \frac{1}{2}\frac{\frac{ha}{3l^3}(a^3 - 2a^2l + l^3) - \frac{l^2i^2e}{l^2 + 16h^2}}{\frac{4h^2l}{15} + \frac{i^2l^2}{8h}\phi_0}W \quad . \quad . \quad (93)$$

Neglecting the effect of axial stress, the terms containing A disappear, and we get,

$$H = \frac{5a(a^3 - 2a^2l + l^3)}{8hl^3}W \quad . \quad . \quad . \quad . \quad (94)$$

49. For *uniform temperature* change t — positive for rise — by making similar substitutions as before in Eq. (89), since

$$\int_0^{l'}\frac{y^2dc}{I} = \frac{8}{15}\frac{h^2l}{I_0},$$

$$\int_0^{l'}\frac{\cos^2\phi\, dc}{A} = \frac{l^2\phi_0}{4hA_0},$$

we get,

$$H_t = \frac{t\theta EI_0}{\frac{8h^2}{15} + \frac{i^2l}{4h}\phi_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

Neglecting axial stress, similarly we get from Eq. (90),

$$H_t = \frac{15\,t\theta I_0E}{8h^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

50. For *change in span length* Δl — negative for increase — we get similarly from Eq. (91),

$$H_\Delta = \frac{EI_0\Delta l}{\frac{8h^2l}{15} + \frac{i^2l^2\phi_0}{4h}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

Neglecting axial stress,

$$H_{\Delta} = \frac{15\, EI_0 \Delta l}{8\, h^2 l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (98)$$

CIRCULAR ARCH WITH TWO HINGES

51. Fig. 42 shows the axis of a symmetrical circular arch-rib with *uniform cross-section.*

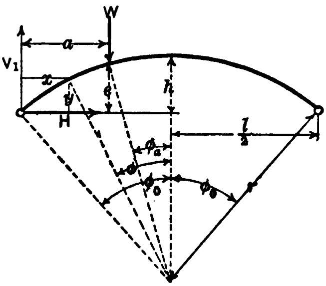

Fig. 42

In Eq. (86), making I and A constants, and putting

$$\frac{I}{A} = i^2,$$

we get

$$H = \frac{1}{2}\,\frac{\int_0^{a'} xydc + a\int_{a'}^{\frac{l'}{2}} ydc - i^2\int_0^a \sin\phi\, dx}{\int_0^{\frac{l'}{2}} y^2dc + i^2\int_0^{\frac{l}{2}} \cos\phi\, dx}\, W.$$

With designations as given in the figure, we have for circular arc,

$$\begin{aligned} x &= r\,(\sin\phi_0 - \sin\phi), \\ dx &= -\, r\cos\phi\, d\phi, \\ y &= r\,(\cos\phi - \cos\phi_0), \\ dc &= \frac{dx}{\cos\phi} = -\, rd\phi, \\ a &= r\,(\sin\phi_0 - \sin\phi_a). \end{aligned}$$

Substituting these in several terms of the expression for H, and integrating,

$$\int_0^{a'} xydc = r^3[\tfrac{1}{2}\sin^2\phi_0 - \sin\phi_0 \sin\phi_a + \tfrac{1}{2}\sin^2\phi_a - (\phi_0 - \phi_a)\sin\phi_0\cos\phi_0 - \cos^2\phi_0 + \cos\phi_0\cos\phi_a],$$

$$a\int_{a'}^{\frac{l'}{2}} ydc = r^3(\sin\phi_0 - \sin\phi_a)(\sin\phi_a - \phi_a\cos\phi_0),$$

$$\int_0^a \sin\phi\, dx = \frac{r}{2}(\sin^2\phi_0 - \sin^2\phi_a),$$

$$\int_0^{\frac{l'}{2}} y^2dc = r^3(\tfrac{1}{2}\phi_0 - \tfrac{3}{2}\sin\phi_0\cos\phi_0 + \phi_0\cos^2\phi_0),$$

$$\int_0^{\frac{l}{2}} \cos\phi\, dx = \frac{r}{2}(\sin\phi_0\cos\phi_0 + \phi_0),$$

we get,

$$H = \frac{1}{2}\,\frac{2\cos\phi_0(\cos\phi_a + \phi_a\sin\phi_a - \cos\phi_0 - \phi_0\sin\phi_0) + \left(1 - \frac{i^2}{r^2}\right)(\sin^2\phi_0 - \sin^2\phi_a)}{(\phi_0 - 3\sin\phi_0\cos\phi_0 + 2\phi_0\cos^2\phi_0) + \frac{i^2}{r^2}(\sin\phi_0\cos\phi_0 + \phi_0)}\,W. \quad (99a)$$

or, since

$$\sin\phi_0 = \frac{l}{2r}, \qquad \cos\phi_0 = \frac{r-h}{r};$$

$$\sin\phi_a = \frac{l - 2a}{2r}, \qquad \cos\phi_a = \frac{r - h + e}{r};$$

$$\phi_0 = \frac{l'}{2r}, \qquad \phi_a = \frac{l' - 2a'}{2r};$$

$$H = \frac{(r-h)\{2e + \phi_a(l - 2a) - \phi_0 l\} + \left(1 - \frac{i^2}{r^2}\right)(l-a)a}{4(r-h)\{\phi_0(r-h) - l\} + \left(1 + \frac{i^2}{r^2}\right)\{2r^2\phi_0 + (r-h)l\}}\,W \quad . \quad (99b)$$

Neglecting axial compression, we get,

$$H = \frac{\sin^2\phi_0 - \sin^2\phi_a + 2\cos\phi_0(\cos\phi_a + \phi_a\sin\phi_a - \cos\phi_0 - \phi_0\sin\phi_0)}{2(\phi_0 - 3\sin\phi_0\cos\phi_0 + 2\phi_0\cos^2\phi_0)}\,W \quad (100a)$$

or

$$H = \frac{(r-h)\{2e + \phi_a(l - 2a) - \phi_0 l\} + a(l-a)}{2\phi_0\{r^2 + 2(r-h)^2\} - 3l(r-h)}\,W \quad . \quad (100b)$$

52. Temperature Stress.—For uniform temperature change of t degrees, Eq. (89) may be written for constant I and A:

$$H_t = \frac{t\theta l E I}{2\int_0^{\frac{l'}{2}} y^2\, dc + 2i^2 \int_0^{\frac{l}{2}} \cos\phi\, dx}.$$

Introducing in this the integrals already given, we obtain,

$$H_t = \frac{t\theta l E I}{r^3(\phi_0 - 3\sin\phi_0\cos\phi_0 + 2\phi_0\cos^2\phi_0) + i^2 r(\sin\phi_0\cos\phi_0 + \phi_0)} \quad (101a)$$

or

$$H_t = \frac{t\theta l E I}{r\left[\phi_0\{r^2 + 2(r-h)^2 + i^2\} - \frac{1}{2}\left(3 - \frac{i^2}{r^2}\right)(r-h)l\right]} \quad . \quad . \quad . \quad (101b)$$

Neglecting axial stress, we get,

$$H_t = \frac{t\theta l E I}{r^3(\phi_0 - 3\sin\phi_0\cos\phi_0 + 2\phi_0\cos^2\phi_0)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (102a)$$

or

$$H_t = \frac{t\theta l E I}{r[\phi_0\{r^2 + 2(r-h)^2\} - \frac{3}{2} l(r-h)]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (102b)$$

53. Displacement Stress.—For change in span length by Δl—negative for increase of span length—we have from Eq. (91),

$$H_\Delta = \frac{EI\Delta l}{2\int_0^{\frac{l'}{2}} y^2\, dc + 2i^2 \int_0^{\frac{l}{2}} \cos\phi\, dx}.$$

Making the same substitutions as before, we get,

$$H_\Delta = \frac{EI\Delta l}{(\text{denominators same as for } H_t)} \quad . \quad . \quad . \quad (103)$$

SEMICIRCULAR ARCH

54. In this, since

$$\phi_0 = \frac{\pi}{2},$$

we obtain at once from Eq. (99*a*),

$$H = \frac{(r^2 - i^2)\cos^2\phi_a}{\pi\,(r^2 + i^2)} W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

Neglecting axial compression, we get from Eq. (100*a*),

$$H = \frac{\cos^2\phi_a}{\pi} W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

55. For *uniform temperature change t,* we have from Eq. (101*a*),

$$H_t = \frac{2\,t\theta lEI}{\pi r\,(r^2 + i^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

and for the same, by neglecting axial stress,

$$H_t = \frac{2\,t\theta lEI}{\pi r^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

56. For *change in span length* Δl similarly from Eq. (103) referred to Eq. (102*a*), we get,

$$H_\Delta = \frac{2\,EI\Delta l}{\pi r\,(r^2 + i^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

and for the same, with axial stress neglected,

$$H_\Delta = \frac{2\,EI\Delta l}{\pi r^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

FLAT ARCH WITH TWO HINGES

57. When the rise of an arch is very small compared with its span length, we may put without material error,

$$dc = dx.$$

Assuming the cross-section of the arch to be uniform throughout, and putting as before

$$\frac{I}{A} = i^2,$$

Eqs. (86), (89), and (91) may be written as follows:

$$H = \frac{1}{2} \frac{\int_0^a xydx + a\int_a^{\frac{l}{2}} ydx - i^2\int_0^{a'} \sin\phi\, dc}{\int_0^{\frac{l}{2}} y^2\, dx + i^2\int_0^{\frac{l'}{2}} \cos\phi\, dc} W \quad . \quad (110)$$

$$H_t = \frac{t\theta lEI}{\int_0^l y^2\, dx + i^2\int_0^{l'} \cos\phi\, dc} \quad . \quad . \quad . \quad . \quad (111)$$

$$H_\Delta = \frac{EI\Delta l}{\int_0^l y^2\, dx + i^2\int_0^{l'} \cos\phi\, dc} \quad . \quad . \quad . \quad . \quad (112)$$

FLAT PARABOLIC ARCH WITH TWO HINGES

(Uniform Cross-Section)

58. For this, we have but to introduce in Eq. (110) the equation of parabola

$$y = \frac{4h}{l^2} x (l - x),$$

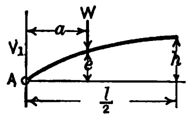

Fig. 43

to obtain expression for H due to any load W (Fig. 43). Integrating the terms of the equation severally, we have,

$$\int_0^a xydx + a\int_a^{\frac{l}{2}} ydx = \frac{a^3h}{3l^2}(4l - 3a) + \frac{ah}{3l^2}(l^3 - 6a^2l + 4a^3),$$

$$\int_0^{a'} \sin\phi\, dc = e,$$

$$\int_0^{\frac{l}{2}} y^2\, dx = \frac{4h^2 l}{15},$$

$$\int_0^{\frac{l}{2}} \cos\phi\, dc = \frac{l}{2},$$

so that

$$H = \frac{1}{2}\frac{\frac{ah}{3\,l^2 i^2}(l - a)(l^2 + al - a^2) - e}{\frac{4\,h^2 l}{15\,i^2} + \frac{l}{2}} W \quad . \quad . \quad . \quad (113)$$

Neglecting e as being inconsiderable in comparison with other terms of the numerator, we obtain,

$$H = \frac{5\,ah\,(l - a)\,(l^2 + al - a^2)}{l^3\,(8\,h^2 + 15\,i^2)} W \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

Further neglecting the effect of axial stress, we get,

$$H = \frac{5\,a\,(l - a)\,(l^2 + al - a^2)}{8\,hl^3} W \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

59. For a *uniformly distributed load* of w per unit length of the span, we obtain, by substituting wda for W in the preceding equations, and integrating between given limits of loading, the equation for H. Thus, referring to Fig. 44, we have, from Eq. (115),

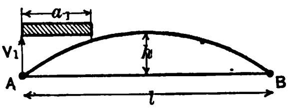

Fig. 44

$$H = \int_0^{a_1} \frac{5\,a\,(l-a)\,(l^2+al-a^2)}{8\,hl^3} wda = \frac{a_1^2\,(5\,l^3 - 5\,a_1^2 l + 2\,a_1^3)\,w}{16\,hl^3} \cdot (116)$$

By taking moment at B,

$$V_1 = \frac{a_1\,(2\,l - a_1)\,w}{2\,l}.$$

For full uniform load,

$$H = \frac{l^2 w}{8\,h} \text{ approximately.}$$

60. Temperature Stress. — Introducing in Eq. (111) the equation of parabola, and integrating as before, we obtain,

$$H_t = \frac{t\theta lEI}{\frac{8\,h^2 l}{15} + i^2 l} = \frac{15\,t\theta EI}{8\,h^2 + 15\,i^2} \quad . \quad . \quad (117)$$

Neglecting axial stress, we get,

$$H_t = \frac{15\, t\theta EI}{8\, h^2} \quad \ldots \ldots \ldots \quad (118)$$

61. Displacement Stress. — For a change of Δl in span length — negative for increase of the latter — similarly we get from Eq. (112),

$$H_\Delta = \frac{15\, EI\Delta l}{l\,(8\, h^2 + 15\, i^2)} \quad \ldots \ldots \quad (119)$$

and for the same with axial stress neglected,

$$H_\Delta = \frac{15\, EI\Delta l}{8\, h^2 l} \quad \ldots \ldots \ldots \quad (120)$$

FLAT CIRCULAR ARCH WITH TWO HINGES

62. Since a circular arc with comparatively small versed sine closely follows parabolic curve, the formulas deduced for parabolic arches (Eqs. 113–120) may be used for this kind of arches without appreciable error.

SPANDREL-BRACED ARCH WITH TWO HINGES

63. The foregoing formulas are generally inapplicable to a spandrel-braced arch, owing to the lack in the latter of definite form in its axis and the irregular variation of the moment of inertia of its section. As the only statically-indeterminate force in this case is again H, in order to find the value of the latter which will make ω a minimum, it is simply necessary to find stresses in each member in terms of H and other external forces, and obtain,

$$\omega = \sum \frac{S^2 L}{2\, AE},$$

extending the second member over the whole structure.

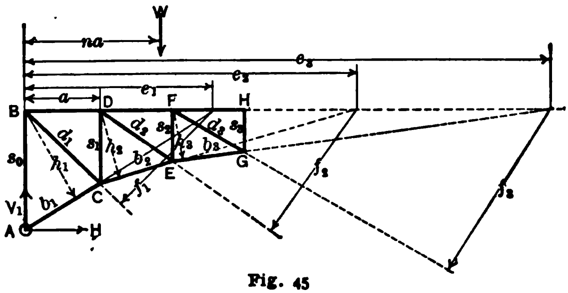

Fig. 45

Fig. 45 shows the left half of a symmetrical spandrel-braced arch. The following designations will be used:

s_0, s_1, etc. . . . the lengths of vertical members.
d_1 d_2, etc. . . . the lengths of diagonal members.
a . . . . . . the horizontal panel length.
b_1 b_2, etc.. . . . the lengths of lower chord-members.
A . . . . . . the sectional area of a member, with suffix corresponding to the members to which it pertains.

If we assume, in the first place, the arch to be loaded with two equal loads of W each, distant na from each end, we obtain the following stresses for the case $n = 1$ by taking moments at successive sections, the arm-lengths being designated as shown in the figure:

WEB-MEMBERS

$$\overline{AB} = -W + \frac{s_0 - s_1}{a} H. \qquad \overline{BC} = \frac{We_1 - Hs_0}{f_1}.$$

$$\overline{CD} = -\frac{We_2 - Hs_0}{e_2 - a}. \qquad \overline{DE} = \frac{Wa - Hs_0}{f_2}.$$

$$\overline{EF} = -\frac{Wa - Hs_0}{e_3 - 2a}. \qquad \overline{FG} = \frac{Wa - Hs_0}{f_3}.$$

CHORD-MEMBERS

$$\overline{BD} = -\frac{Wa - H(s_0 - s_1)}{s_1}. \qquad \overline{AC} = -\frac{Hs_0}{h_1}.$$

$$\overline{DF} = -\frac{Wa - H(s_0 - s_2)}{s_2}. \qquad \overline{CE} = \frac{Wa - Hs_0}{h_2}.$$

$$\overline{FH} = -\frac{Wa - H(s_0 - s_3)}{s_3}. \qquad \overline{EG} = \frac{Wa - Hs_0}{h_3}.$$

Introducing these in the expression for total internal work, we get,

$$\frac{\Sigma S^2 L}{2AE} = \frac{2}{E}\Bigg\{\left(-W + \frac{s_0 - s_1}{a}H\right)^2 \frac{s_0}{2A_{ab}} + \frac{(We_2 - Hs_0)^2}{(e_2 - a)^2}\frac{s_1}{2A_{cd}}$$

$$+\left(\frac{Wa - Hs_0}{e_3 - 2a}\right)^2 \frac{s_2}{2A_{ef}} + \left(\frac{We_1 - Hs_0}{f_1}\right)^2 \frac{d_1}{2A_{bc}} + \left(\frac{Wa - Hs_0}{f_2}\right)^2 \frac{d_2}{2A_{de}}$$

$$+\left(\frac{Wa - Hs_0}{f_3}\right)^2 \frac{d_3}{2A_{fg}} + \left(\frac{Hs_0}{h_1}\right)^2 \frac{b_1}{2A_{ac}} + \left(\frac{Wa - Hs_0}{h_2}\right)\frac{b_2}{2A_{ce}}$$

$$+\left(\frac{Wa - Hs_0}{h_3}\right)^2 \frac{b_3}{2A_{eg}} + \left(\frac{Wa - H(s_0 - s_1)}{s_1}\right)^2 \frac{a}{2A_{bd}}$$

$$+\left(\frac{Wa - H(s_0 - s_2)}{s_2}\right)^2 \frac{a}{2A_{df}} + \left(\frac{Wa - H(s_0 - s_3)}{s_3}\right)^2 \frac{a}{2A_{fh}}\Bigg\}.$$

Differentiating this with respect to H, and setting the differential coefficient equal to zero, we at once get,

$$H = \frac{\dfrac{(s_0 - s_1)s_0}{aA_{ab}} + \dfrac{e_2 s_0 s_1}{(e_2 - a)^2 A_{cd}} + \dfrac{a s_0 s_2}{(e_3 - 2a)^2 A_{ef}} + \dfrac{e_1 s_0 d_1}{f_1^2 A_{bc}} + \dfrac{a^2(s_0 - s_1)}{s_1^2 A_{bd}}}{\dfrac{(s_0 - s_1)^2 s_0}{a^2 A_{ab}} + \dfrac{s_0^2 s_1}{(e_2 - a)^2 A_{cd}} + \dfrac{s_0^2 s_2}{(e_3 - 2a)^2 A_{ef}} + \dfrac{b_1 s_0^2}{h_1^2 A_{ac}} + \dfrac{d_1 s_0^2}{f_1^2 A_{bc}}}$$

$$\frac{+\dfrac{a s_0 b_2}{h_2^2 A_{ce}} + \dfrac{a s_0 d_2}{f_2^2 A_{de}} + \dfrac{a^2(s_0 - s_2)}{s_2^2 A_{df}} + \dfrac{a s_0 b_3}{h_3^2 A_{eg}} + \dfrac{a s_0 d_3}{f_3^2 A_{fg}} + \dfrac{a^2(s_0 - s_3)}{s_3^2 A_{fh}}}{+\dfrac{a(s_0 - s_1)^2}{s_1^2 A_{bd}} + \dfrac{b_2^2 s_0^2}{h_2^2 A_{ce}} + \dfrac{s_0^2 d_2}{f_2^2 A_{de}} + \dfrac{a(s_0 - s_2)^2}{s_2^2 A_{df}} + \dfrac{b^3 s_0^2}{h_3^2 A_{eg}} + \dfrac{d_3 s_0^2}{f_3^2 A_{fg}} + \dfrac{a(s_0 - s_3)^2}{s_3^2 A_{fh}}} W,$$

which is the value of H for 2 W, so that H for 1 W will be one-half the amount given by this equation.

In a similar manner we obtain in the arch of this type with *any number of panels*, for 1 W the following expression for H:

$$H = \frac{\sum_{0}^{na}\left\{\frac{pa^2(s_0-s)}{s^2A_u}+\frac{pas_0b}{h^2A_l}+\frac{es_0s}{(e-pa)^2A_v}+\frac{es_0d}{f^2A_d}\right\}+\frac{(s_0-s_1)s_0}{aA_{ab}}}{2\sum_{0}^{\frac{l}{2}}\left\{\frac{(s_0-s)^2a}{s^2A_u}+\frac{s_0{}^2b}{h^2A_l}+\frac{s_0{}^2s}{(e-pa)^2A_v}\right.}$$

$$\frac{+\sum_{na}^{\frac{l}{2}}\left\{\frac{na^2(s_0-s)}{s^2A_u}+\frac{nas_0b}{h^2A_l}+\frac{nas_0s}{(e-pa)^2A_v}+\frac{nas_0d}{f^2A_d}\right\}}{\left.+\frac{s_0{}^2d}{f^2A_d}\right\}+2\,\frac{(s_0-s_1)^2s_0}{a^2A_{ab}}}\,W \,. \qquad (121)$$

in which p represents, in case of chord-members, the distance — in number of panels — of the panel point opposite the member under consideration, and in case of web-members the ordinal number — from the nearest support — of the upper chord opposite the web-member in question. Thus, referring to Fig. 45, we find for DF, EG, EF, and DE, $p = 2$.

A_u, A_l, A_v, and A_d = cross-sectional areas of the upper and lower chords, verticals and diagonals respectively.

e . . . . = distance from the support — nearest to the member — of the intersection of the lower chord-member opposite the web under consideration with the upper chord.

It is further understood that Σ is to be extended over the loaded half of the arch only.

64. Eq. (121) gives mathematically correct results so long as the supports are perfectly immovable, and is applicable when dimensions of all the members of the arch are given. For designing an arch of this kind, the calculation may be started with following approximations:

Assume each chord to be of uniform cross-section throughout its length, and let

$$m = \frac{A_l}{A_u}.$$

Further, neglect the effect of web-stresses, which is generally inconsiderable when compared with that of chord-stresses. Then we get from (121) the following approximate expression for H, freed of all the cross-sectional areas of members:

$$H = \frac{\sum_{0}^{na} \left\{ \frac{pa^2 (s_0 - s)}{s^2} + \frac{pas_0 b}{mh^2} \right\} + \sum_{na}^{\frac{l}{2}} \left\{ \frac{na^2 (s_0 - s)}{s^2} + \frac{nas_0 b}{mh^2} \right\}}{2 \sum_{0}^{\frac{l}{2}} \left\{ \frac{s_0^2 b}{mh^2} + \frac{(s_0 - s)^2 a}{s^2} \right\}} W. \quad (122)$$

With H obtained with this equation, the dimensions of all the members may be calculated, and then corrected, if desired, by the use of Eq. (121). Generally, Eq. (122) by itself gives results sufficiently correct for all practical purposes.

65. Temperature Stress. — The internal work caused by H_t in the arch of Fig. 45, neglecting the effect of web-stresses, will be,

$$\omega = \frac{H_t^2}{EA} \left\{ \frac{s_0^2 b_1}{mh_1^2} + \frac{s_0^2 b_2}{mh_2^2} + \frac{s_0^2 b_3}{mh_3^2} + \frac{(s_0 - s_1)^2 a}{s_1^2} + \frac{(s_0 - s_2)^2 a}{s_2^2} + \frac{(s_0 - s_3)^2 a}{s_3^2} \right\}$$

in which A represents the section of upper chord, and mA that of the lower.

Since

$$\frac{d\omega}{dH_t} = t\theta l,$$

we get,

$$H_t = \frac{t\theta l E A}{2\left\{\frac{s_0^2 b_1}{mh_1^2} + \frac{s_0^2 b_2}{mh_2^2} + \frac{s_0^2 b_3}{mh_3^2} + \frac{(s_0 - s_1)^2 a}{s_1^2} + \frac{(s_0 - s_2)^2 a}{s_2^2} + \frac{(s_0 - s_3)^2 a}{s_3^2}\right\}},$$

t being as before positive for rising temperature. Generally, for any number of panels we get in a similar manner,

$$H_t = \frac{t\theta l E A}{2\sum_0^{\frac{l}{2}} \left\{\frac{s_0^2 b}{mh^2} + \frac{(s_0 - s)^2 a}{s^2}\right\}} \quad . \quad . \quad . \quad (123)$$

66. Displacement Stress. — From the preceding discussions, it will at once be seen that for a change in span length, of Δl, we have but to substitute Δl — negative for increase of span length — for $t\theta l$ to obtain an expression for H_Δ, so that we get,

$$H_\Delta = \frac{EA\Delta l}{2\sum_0^{\frac{1}{2}l} \left\{\frac{s_0^2 b}{mh^2} + \frac{(s_0 - s)^2 a}{s^2}\right\}} \quad . \quad . \quad . \quad (124)$$

THE STRESSES IN FLANGES AND WEBS OF A RIB

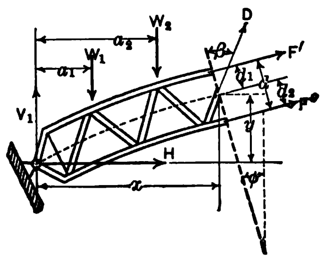

Fig. 46

67. Knowing V_1 and H for a given loading, the stresses in the flanges and web of a parallel rib may at once be obtained statically. Thus, at a radial section through any point x, y (Fig. 46) of an arch-rib, let

F' = the upper flange stress.
F'' = the lower flange stress.
D = web-stress.
$d = d_1 + d_2$ = distance between centres of gravity of upper and lower flanges.

Taking moment with respect to the point x, y in the axis of the rib, we have,

$$V_1 x - Hy - \sum_0^x W(x-a) + F'd_1 - F''d_2 = 0,$$

from which

$$F''d_2 - F'd_1 = V_1 x - Hy - \sum_0^x W(x-a).$$

But

$$F'' + F' = N = -(V \sin\phi + H \cos\phi) \text{ (Art. 43).}$$

Combining these two equations, we get,

$$F'' = \frac{1}{d}\left\{V_1 x - Hy - \sum_0^x W(x-a) - (V \sin\phi + H\cos\phi)d_1\right\} \quad (125)$$

$$F' = -\frac{1}{d}\left\{V_1 x - Hy - \sum_0^x W(x-a) + (V \sin\phi + H\cos\phi)d_2\right\} \quad (126)$$

If

$$d_1 = d_2,$$

i.e., if the section of the rib is symmetrical about the neutral axis,

$$F'' = \frac{1}{d}\left\{V_1 x - Hy - \sum_0^x W(x-a)\right\} - \frac{d}{2}(V\sin\phi + H\cos\phi)\Big\} \quad (127)$$

$$F' = -\frac{1}{d}\left\{V_1 x - Hy - \sum_0^x W(x-a)\right\} + \frac{d}{2}(V\sin\phi + H\cos\phi)\Big\} \quad (128)$$

In case the chord-members are *curved between two panel points*, the direct stress thus found must be combined with the moment equal to the direct stress multiplied by the versed sine of the panel arc.

Again, since at the section

$$D \cos \beta = T = -(V \cos \phi - H \sin \phi) \text{ (Art. 43)},$$
$$D = -(V \cos \phi - H \sin \phi) \sec \beta \quad . \quad . \quad . \quad (129)$$

in which β represents the inclination of the web-member to the radius of the arc at the section.

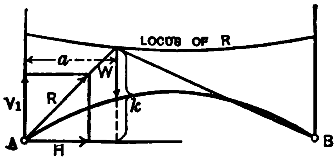

Fig. 47

In case the chords are *not parallel*, the stress in any member is best obtained by taking moment with respect to the intersection of the other two belonging to the panel, as in the case of spandrel-braced arch.

POSITION OF LOADS FOR MAXIMUM STRESS

68. For finding the position of loads to give maximum stress at any point of the arch, *reaction locus* may be made use of with advantage.

Let

R = reaction at A due to any load W.
V_1 = vertical component of R.
H = horizontal component of R.
k = ordinate to the locus of R.

Referring then to Fig. 47, it will be seen that,

$$k = \frac{V_1}{H} a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

which makes the locus at once determinate.

69. Having the locus drawn, the mode of loading giving maximum moment with respect to any given point of the rib may be laid off, remembering that R passing above the point produces + moment, and that below, − moment. Thus in Fig. 48 it will be seen that the stress in any member EF will be maximum when the moment with respect to point O reaches its greatest amount. The curve $CKGD$ being the reaction locus, a load at G will produce no stress in EF, for then the reaction passes through O. For loads to the right of G, by considering the portion of the rib left of O, it will be seen that the moment of R being −, the stress in EF will be tension, while for loads to the left of G, by considering successively the left and right portions of the rib, the moment with respect to O being positive, EF will be in compression. For loads beyond K, considering the right portion of the arch, the moment of right reaction with respect to O being negative, EF will once more be in tension.

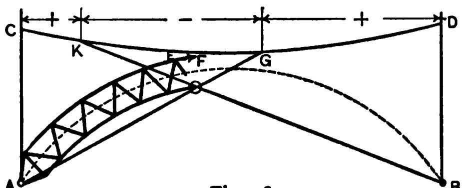

Fig. 48

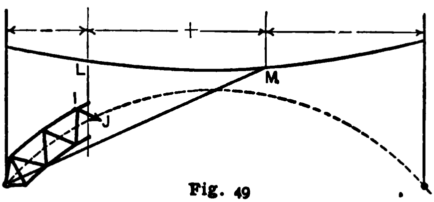

Fig. 49

Similarly, for maximum stress in a lower chord-member, the reaction line drawn through the panel point opposite the member will give the mode of loading.

70. For maximum stress in a web-member generally, the reaction line is to be passed through the intersection of chord-members belonging to the panel to determine the limits of loading. In case the chords are parallel at the panel, since the web-member will not then be strained when the direction of the resultant force coincides with that of the chords, the position of load, for no stress in the web-member, is given by drawing the reaction line parallel to the chords, or to the tangent to the curve. Thus in Fig. 49, point M is the position of load producing

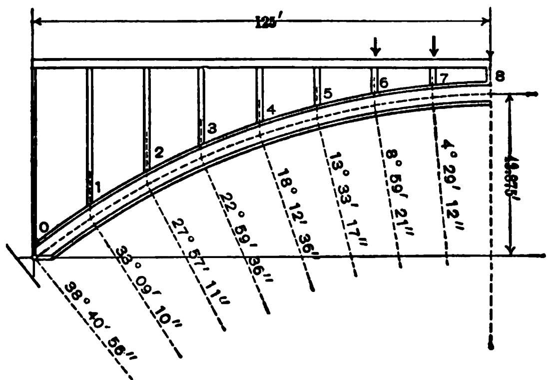

Fig. 50

no stress in IJ. Another position of load for no stress in IJ is at L directly over the section, — a point which will be evident by considering alternately the right and left of the section with its respective reactions. The signs of shear, and, with them, those of the stress in the web-member due to loads between the points of no stress, will at once be known by referring to Art. 43.

Example. — In the circular plate-webbed arch with two hinges, of which Fig. 50 shows its left half, to calculate the maximum stress in the rib at panel point 3, due to following panel loads:

Dead load = 20 tons per panel,
Live load = 10 tons per panel.

The following dimensions are given:

$l = 250$ ft.
$r = 200$ ft.
$\phi_0 = 38^\circ - 40' - 56'' = .67514$.
Panel length = 15.625 ft.
Cross-section uniform and symmetrical throughout,
$\frac{i^2}{r^2} = .00019$.
Effective depth (dist. of c. g. of flanges) = 6 ft.

The distances of the points of application of loads, etc., are as follows:

a (ft.).	e (ft.).	ϕ_a (circ. meas.).	$\phi_a (l - 2a)$.	$a (l - a)$.	$\left(1 - \frac{i^2}{r^2}\right) a(l - a)$.
15.63	11.32	.57863	126.58	3662	3661
31.25	20.54	.48788	91.48	6836	6835
46.88	27.99	.40114	62.68	9517	9520
62.50	33.86	.31782	39.73	11719	11716
78.13	38.31	.23658	22.18	13428	13425
93.75	41.42	.15689	9.81	14648	14646
109.38	43.26	.07831	2.45	15381	15378
125.00	43.88	.00000	0.00	15625	15622

In Eq. (99*b*),

$$H = \frac{(r - h)\{2e + \phi_a(l - 2a) - \phi_0 l\} + \left(1 - \frac{i^2}{r^2}\right)(l - a)a}{4(r - h)\{\phi_0(r - h) - l\} + \left(1 + \frac{i^2}{r^2}\right)\{2r^2\phi_0 + (r - h)l\}} W.$$

Introducing the numerical values in the numerator, the denominator being

$$4(r-h)\{\phi_0(r-h)-l\} + \left(1+\frac{i^2}{r^2}\right)\{2r^2\phi_0 + (r-h)l\} = 2766,$$

we get for $W = 1$ the following values of H:

Load at	1	2	3	4	5	6	7	8
$H =$	.2189	.4263	.6118	.7737	.9027	.9971	1.0546	1.0739

For the dead load we have then,

$V_1 = 7\frac{1}{2} \times 20 = 150$ tons,

$H = 2 \times 20\,(.2189 + .4263 + .6118 + .7737 + .9027 + .9971 + 1.0546 + .5370) = 220.88$ tons.

Drawing the reaction locus, and passing reaction lines through E and F,—the centres of gravity of upper and lower

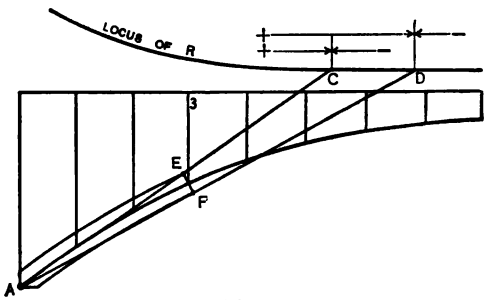

Fig. 51

flanges at 3,—we find (Fig. 51) that all loads lying to the right of C produce negative moment with respect to E and hence compression in the lower flange; while for similar reason all loads to the left of D produce compression in the upper flange.

By drawing AM parallel to the tangent to the axis of the rib at 3, and passing a vertical through E, we find that the loads between L and M produce positive shear in the section EF, and those outside of LM, the negative shear.

The amounts of H for the positions of the live load found above are:

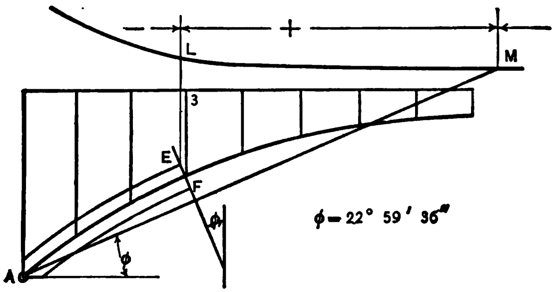

Fig. 52

For the live load covering $A - D$,

$$V_1 = \tfrac{10}{16}(15 + 14 + 13 + 12 + 11 + 10) = 46.88 \text{ tons.}$$
$$H = 10\,(.2189 + \cdots .9027 + 1.0546) = 39.88 \text{ tons.}$$

For the live load covering $C - B$,

$$V_1 = \tfrac{10}{16}(1 + \cdots 10) = 34.38 \text{ tons.}$$
$$H = 10\{2\,(1.0739 + 1.0546 + .9971) + .9027 + \cdots .2189\}$$
$$= 91.85 \text{ tons.}$$

For the live load covering $A - L$ and $M - B$,

$$V_1 = \tfrac{10}{16}(1 + \cdots 7 + 14 + 15) = 35.62 \text{ tons.}$$
$$H = 10\{2(.2189 + .4263) + .6118 + \cdots 1.0546\} = 56.30 \text{ tons.}$$

For the live load covering $L - M$,

$$V_1 = \tfrac{10}{16}(8 + 9 + 10 + 11 + 12 + 13) = 39.38 \text{ tons.}$$
$$H = 10\,(.6118 + \cdots 1.0739) = 54.14 \text{ tons.}$$

At the neutral axis under the panel point 3, we have

$x = 46.88$ ft. $y = 27.99$ ft. $\phi = 22°59'36''$.
$\sin\phi = .3906$. $\cos\phi = .9336$.

Substituting these values in Eqs. (127) and (128), we get for the total stress in flanges at 3:

$$F'' = \tfrac{1}{8}\{184.38 \times 46.88 - 312.73 \times 27.99 - 20(31.25 + 15.63) - 3(184.38 \times .3906 + 312.73 \times .9336)\} = -356.38 \text{ tons.}$$

$$F' = -\tfrac{1}{8}\{196.88 \times 46.88 - 260.76 \times 27.99 - 30(31.25 + 15.63) + 3(196.88 \times .3906 + 260.76 \times .9336)\} = -247.62 \text{ tons.}$$

As to shear acting in the normal section at 3, we have (Art. 43),

$$V\cos\phi - H\sin\phi$$
$$= (150 + 39.38 - 2 \times 20).9336 - (220.88 + 54.14).3906$$
$$= 32.04 \text{ tons for maximum.}$$
$$= (150 + 35.62 - 2 \times 30).9336 - (220.88 + 56.30).3906$$
$$= 9.01 \text{ tons for minimum.}$$

Had we neglected the effect of axial stress in the preceding calculation, Eq. (100*b*) would have given for *H*,

Load at	1	2	3	4	5	6	7	8
$H=$	.2206	.4295	.6164	.7794	.9095	1.0046	1.0625	1.0811

So that we get for dead load,

$$V_1 = 150 \text{ tons,}$$
$$H = 20 \times 2 \times 5.5630 = 222.52 \text{ tons;}$$

and for live load covering *A* -- *D*,

$$V_1 = 46.88 \text{ tons,}$$
$$H = 39.60.$$

Substituting these values in Eq. (128), we get for maximum compression in the upper flange at 3,

$$F' = -\tfrac{1}{8}\{196.88 \times 46.88 - 262.12 \times 27.99 - 30\,(31.25 + 15.63) + 3\,(196.88 \times .3906 + 262.12 \times .9336)\} = -241.91 \text{ tons.}$$

It will be seen from these calculations, that the effect of neglecting axial stress, while producing a difference of less than 0.8 per cent in the amounts of H, is more strongly felt in chord-stresses, in which the difference, in the case taken, amounts to more than 2 per cent.

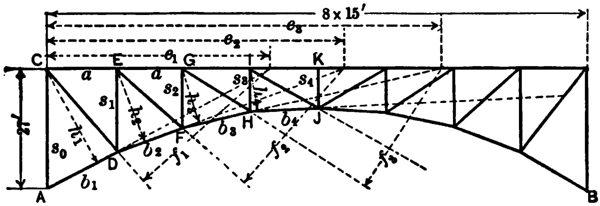

Fig. 53

Example. — In the spandrel-braced arch of Fig. 53 with dimensions as tabulated below, to calculate the maximum stress in a member FH due to the following panel loads:

Dead load . . 16 tons per panel,
Live load . . 20 tons per panel.

Lengths of members and arms in feet:

	s.	*d.*	*b.*	*e.*	*h.*	*f.*
1	18.7	23.9	17.1	48.8	23.6	38.2
2	13.2	20.0	16.0	66.0	17.6	33.7
3	10.0	18.0	15.3	91.9	12.9	34.4
4	9.0	17.5	15.0	195.0	10.0	77.1

Assuming $m = 2$ (Art. 64), we have in Eq. (122):

Load	Panel.	$\sum_0^{na} \frac{pa^2(s_0-s)}{s^2}$	$\sum_{na}^{\frac{l}{2}} \frac{na^2(s_0-s)}{s^2}$	$\sum_0^{na} \frac{pa\,s_0 b}{2h^2}$	$\sum_{na}^{\frac{l}{2}} \frac{na\,s_0 b}{mh^2}$	$\sum_0^{\frac{l}{2}} \frac{s_0^2 b}{mh^2}$	$\sum_0^{\frac{l}{2}} \frac{(s_0-s)^2 a}{s^2}$
Load 1 at $E (n=1)$	CE	5.34	. . .	. . .	. . .	11.19	2.96
	EG	. . .	17.82	. . .	10.46	19.37	16.40
	GI	. . .	38.25	. . .	18.62	33.51	43.35
	IK	. . .	50.00	. . .	30.38	54.68	60.00
	Σ	5.34	106.07	. . .	59.46	118.75	122.71
	$H_1 = \frac{5.34 + 106.07 + 59.46}{2(118.75 + 122.71)} = .356$						
Load 1 at $G (n=2)$	CE	5.34	. . .	. . .	. . .	. . .	. . .
	EG	35.64	. . .	10.46	. . .	. . .	. . .
	GI	. . .	76.50	. . .	37.24	. . .	. . .
	IK	. . .	100.00	. . .	60.76	. . .	. . .
	Σ	40.98	176.50	10.46	98.00	118.75	122.71
	$H_2 = \frac{325.95}{482.92} = .675$						
Load 1 at $I (n=3)$	CE	5.34	. . .	. . .	. . .	. . .	. . .
	EG	35.64	. . .	10.46	. . .	. . .	. . .
	GI	114.75	. . .	37.24	. . .	. . .	. . .
	IK	. . .	150.00	. . .	91.14	. . .	. . .
	Σ	155.73	150.00	47.70	91.14	118.75	122.71
	$H_3 = \frac{444.57}{482.92} = .921$						
Load 1 at $K (n=4)$	CE	5.34	. . .	. . .	. . .	. . .	. . .
	EG	35.64	. . .	10.46	. . .	. . .	. . .
	GI	114.75	. . .	37.24	. . .	. . .	. . .
	IK	200.00	. . .	91.14	. . .	. . .	. . .
	Σ	355.73	. . .	138.84	. . .	118.75	122.71
	$H_4 = \frac{494.57}{482.92} = 1.024$						

For the dead load we then have,

$V_1 = 16 \times 3\frac{1}{2} = 56$ tons,
$H = 16 \times 2(.356 + .675 + .921 + .512) = 78.85$ tons.

Drawing the reaction locus and then the reaction line through G (Fig. 54) to the locus, we see that all loads to the right of L will produce compression in FH, while those to the left, tension.

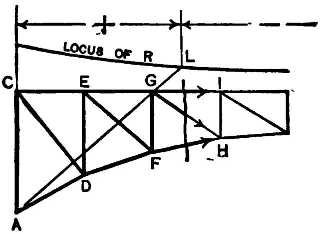

Fig. 54

For these positions of the live load we have,
Live load covering $C - G$,

$V_1 = \frac{20}{8}(7 + 6) = 32.50$ tons,
$H = 20(.356 + .675) = 20.62$ tons.

Live load covering $I - B$,

$V_1 = \frac{20}{8}(1 + 2 + 3 + 4 + 5) = 37.5$ tons,
$H = 20(.356 + .675 + .921 + 1.024 + .921) = 77.94$ tons.

Taking moment with respect to G, we obtain the following extreme stresses in FH:

$$\text{Max. } FH = \frac{1}{12.9}\{(56 + 37.5)\,30 - (78.85 + 77.94)\,27 - 16 \times 15\}$$
$$= -\ 129.33 \text{ tons.}$$

$$\text{Min. } FH = \frac{1}{12.9}\{(56 + 32.5)\,30 - (78.85 + 20.62)\,27 - 16 \times 15\}$$
$$= 3 -\ 43.47 \text{ tons.}$$

Suppose now that the following cross-sections (in sq. in.) of the members are given:

Panel.	Upper Chord.	Lower Chord.	Diagonals.	Verticals.
CE	6	28	7	6
EG	9	23	7	10
GI	15	17	7	9
IK	17	17	5	7

Then in Eq. (121),

$H =$

$$\frac{\sum_{0}^{na}\left\{\frac{pa^2(s_0-s)}{s^2A_u}+\frac{pas_0b}{h^2A_l}+\frac{es_0s}{(e-pa)^2A_v}+\frac{es_0d}{f^2A_d}\right\}+\frac{(s_0-s_1)s_0}{aA_{ab}}+\sum_{na}^{\frac{l}{2}}\left\{\frac{na^2(s_0-s)}{s^2A_u}+\frac{nas_0b}{h^2A_l}+\frac{nas_0s}{(e-pa)^2A_v}+\frac{nas_0d}{f^2A_d}\right\}}{2\sum_{0}^{\frac{l}{2}}\left\{\frac{(s_0-s)^2a}{s^2A_u}+\frac{s_0{}^2b}{h^2A_l}+\frac{s_0{}^2s}{(e-pa)^2A_v}+\frac{s_0{}^2d}{f^2A_d}\right\}+2\frac{(s_0-s_1)^2s_0}{a^2A_{ab}}}W$$

we have,

For Denominator

Panel	$\sum_{0}^{\frac{l}{2}} \frac{a\,(s_0-s)^2}{s^2A_u}$	$\sum_{0}^{\frac{l}{2}} \frac{s_0{}^2b}{h^2A_l}$	$\sum_{0}^{\frac{l}{2}} \frac{s_0{}^2s}{(e-pa)^2A_v}$	$\sum_{0}^{\frac{l}{2}} \frac{s_0{}^2d}{f^2A_d}$	$\frac{(s_0-s_1)^2s_0}{a^2A_{ab}}$
CE	.49	.80	1.19	1.70	. . .
EG	1.82	1.64	.83	1.83	. . .
GI	2.89	3.94	.47	1.58	. . .
IK	3.53	6.43	. . .	.43	. . .
Σ	8.73	12.81	2.49	5.54	1.38

Denominator = 2 (8.73 + 12.81 + 2.49 + 5.54 + 1.38) = 61.90.

For Numerator

	Panel.	$\sum_{0}^{na} \frac{pa^2(s_0-s)}{s^2 A w}$	$\sum_{na}^{l} \frac{na^2(s_0-s)}{s^2 A w}$	$\sum_{0}^{na} \frac{pas_0 b}{k^2 A i}$	$\sum_{na}^{l} \frac{nas_0 b}{k^2 A i}$	$\sum_{0}^{na} \frac{es_0 s}{(e-pa)^2 A v}$	$\sum_{na}^{l} \frac{nas_0 s}{(e-pa)^2 A v}$	$\sum_{0}^{na} \frac{es_0 d}{f^2 A d}$	$\sum_{na}^{l} \frac{nas_0 d}{f^2 A d}$	$\frac{(s_0-s_1)s_0}{a A m}$
$n = 1$	*CE*	.89	. . .	. . .	. . .	2.16	. . .	3.10	. . .	. . .
	EG	. . .	1.98	. . .	.91	. . .	.46	. . .	1.02	. . .
	GI	. . .	2.55	. . .	2.19	. . .	.26	. . .	.88	. . .
	KI	. . .	2.94	. . .	3.57	. . .	. . .	. . .	.24	. . .
	Σ	.89	7.47	. . .	6.67	2.16	.72	3.10	2.14	2.49
	$H_1 = \frac{25.64}{61.90} = .414$									
$n = 2$	*CE*	.89	. . .	. . .	. . .	2.16	. . .	3.10	. . .	. . .
	EG	3.96	. . .	.91	. . .	2.02	. . .	4.48	. . .	. . .
	GI	. . .	5.10	. . .	4.38	. . .	.52	. . .	1.76	. . .
	IK	. . .	5.88	. . .	7.14	. . .	. . .	. . .	.48	. . .
	Σ	4.85	10.98	.91	11.52	4.18	.52	7.58	2.24	2.49
	$H_2 = \frac{45.27}{61.90} = .732$									
$n = 3$	*CE*	.89	. . .	. . .	. . .	2.16	. . .	3.10	. . .	. . .
	EG	3.96	. . .	.91	. . .	2.02	. . .	4.48	. . .	. . .
	GI	7.65	. . .	4.38	. . .	1.61	. . .	5.40	. . .	. . .
	IK	. . .	8.82	. . .	10.71	. . .	. . .	. . .	.72	. . .
	Σ	12.50	8.82	5.29	10.71	5.79	. . .	12.98	.72	2.49
	$H_3 = \frac{59.30}{61.90} = .958$									
$n = 4$	*CE*	.89	. . .	. . .	. . .	1.28	. . .	3.10	. . .	. . .
	EG	3.96	. . .	.91	. . .	.94	. . .	4.48	. . .	. . .
	GI	7.65	. . .	4.38	. . .	.35	. . .	5.40	. . .	. . .
	IK	11.76	. . .	10.71	. . .	. . .	. . .	3.11	. . .	. . .
	Σ	24.26	. . .	16.00	. . .	2.57	. . .	16.09	. . .	2.49
	$H_4 = \frac{61.41}{61.90} = .992$									

With these values of H, we get for dead load,

$$H = 16 \times 2\,(.414 + .732 + .958 + .496) = 83.20 \text{ tons},$$

$$V_1 = 3\tfrac{1}{2} \times 16 = 56 \text{ tons}.$$

For live load covering $I - B$,

$$H = 20\,(.414 + .732 + .958 + .992 + .958) = 81.08 \text{ tons},$$

$$V_1 = \tfrac{20}{8} \times 15 = 37.5 \text{ tons}.$$

Taking moment at G as before, we get for the maximum stress in FH,

$$\frac{1}{12.9}\{(56 + 37.5)\,30 - (83.20 + 81.08)\,27 - 16 \times 15\} = -144.96 \text{ tons}.$$

Comparing the values of H obtained by the use of Eqs. (121) and (122), it will be seen that the neglect of web-stresses and the assumption of uniform chord sections have led to no appreciable error, the difference being about $4\tfrac{1}{2}$ per cent; but its effect is more strongly felt by individual members, as shown by a comparison of maximum stresses in EF, the difference of the latter amounting to more than 10 per cent.

BALANCED ARCH WITH TWO HINGES

71. In a balanced arch, such as shown in Fig. 55, with independent span at each end, the method of calculating reactions does not differ in general from that explained in the case of spandrel-braced arch, the main difference being that in the present case V_1 and H will be + or − according to modes of loading.

EXAMPLE. — In the symmetrical balanced arch of Fig. 55, to calculate H, V_1 and V_2 due to a uniform load of 1.5 tons per ft. run covering the whole of the left arm.

Since the left cantilever arm is loaded at its end with

$$45 + 22.5 = 67.5^t,$$

by taking moments at A and B we get,

$$-67.5 \times 60 - 45 \times 30 - V_2 \times 240 = 0. \qquad V_2 = -22.5^t.$$
$$-67.5 \times 300 - 45 \times 270 + V_1 \times 240 = 0. \qquad V_1 = +135.0^t.$$

Neglecting the stresses in web-members, we have — by taking moments at successive sections — the following stresses, which, with given lengths and cross-sectional areas of members, give the corresponding works of resistance in chord-members:

	Panel.	S. (tons.)	A. (sq. in.)	L. (ft.)	$\frac{S^2L}{2AE}$.
Upper Chord	I	$102.7 + \frac{9}{23}H$	20	30	$\left(102.7 + \frac{9}{23}H\right)^2 \frac{30}{40E}$
	II	$119.1 + \frac{15}{17}H$	25	30	$\left(119.1 + \frac{15}{17}H\right)^2 \frac{30}{50E}$
	III	$130.0 + \frac{19}{13}H$	28	30	$\left(130.0 + \frac{19}{13}H\right)^2 \frac{30}{56E}$
	IV, V	$112.5 + \frac{5}{3}H$	30	60	$\left(112.5 + \frac{5}{3}H\right)^2 \frac{60}{60E}$
	VI	$78.0 + \frac{19}{13}H$	28	30	$\left(78.0 + \frac{19}{13}H\right)^2 \frac{30}{56E}$
	VII	$40.0 + \frac{15}{17}H$	25	30	$\left(40.0 + \frac{15}{17}H\right)^2 \frac{30}{50E}$
	VIII	$14.7 + \frac{9}{23}H$	20	30	$\left(14.7 + \frac{9}{23}H\right)^2 \frac{30}{40E}$
Lower Chord	I	$-\left(98.2 + \frac{64}{55}H\right)$	55	35	$\left(98.2 + \frac{64}{55}H\right)^2 \frac{35}{110E}$
	II	$-\left(112.5 + \frac{32}{21}H\right)$	55	33	$\left(112.5 + \frac{32}{21}H\right)^2 \frac{33}{110E}$
	III	$-(126.6 + 2H)$	52	31	$\left(126.6 + 2H\right)^2 \frac{31}{104E}$
	IV	$-\left(130.0 + \frac{32}{13}H\right)$	50	30	$\left(130.0 + \frac{32}{13}H\right)^2 \frac{30}{100E}$
	V	$-\left(77.9 + \frac{32}{13}H\right)$	50	30	$\left(77.9 + \frac{32}{13}H\right)^2 \frac{30}{100E}$
	VI	$-(42.2 + 2H)$	52	31	$\left(42.2 + 2H\right)^2 \frac{31}{104E}$
	VII	$-\left(16.1 + \frac{32}{21}H\right)$	55	33	$\left(16.1 + \frac{32}{21}H\right)^2 \frac{33}{110E}$
	VIII	$-\frac{64}{55}H$	55	35	$\left(\frac{64}{55}H\right)^2 \frac{35}{110E}$

Summing up all the terms of the fifth column, and differentiating the sum with respect to H, and setting the differential coefficient equal to zero, we get,

$$H = -56.5^t,$$

showing that the horizontal reaction is directed opposite to that shown by the arrow at A.

For loads in the central span, considering the side

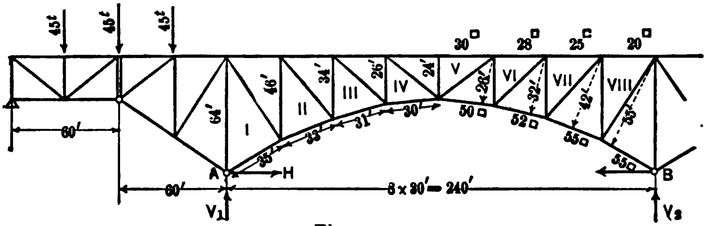

Fig. 55

spans to be weightless, the calculation of H and V is exactly the same as explained in Art. 63.

H will, therefore, be positive — i.e., acting toward right — or negative according as the central or side span is loaded.

TIED ARCH WITH TWO HINGES

72. In the tied arch the horizontal thrust is taken up by the resistance offered by the tie. It has an advantage of the absence of stresses due either to changes of temperature or displacement of supports. Figs. 56 and 57 show arches of this kind. Representing by A_t the cross-section of the tie, we have for the work of resistance in the same due to H,

$$\frac{H^2 l}{2 A_t E}.$$

This, then, is to be included in the expression for the work of resistance.

In the case of arch-rib (Fig. 56), referring to Art. 45, we get, after differentiating ω with respect to H, the following equation of the latter for one load W:

$$H = \frac{1}{2} \frac{\int_0^{a'} \frac{xydc}{I} + a\int_{a'}^{\frac{l'}{2}} \frac{ydc}{I} - \int_0^{a} \frac{\sin\phi dx}{A}}{\int_0^{\frac{l'}{2}} \frac{y^2dc}{I} + \int_0^{\frac{l}{2}} \frac{\cos\phi\, dx}{A} + \frac{l}{2A_t}} W \quad . \quad (131)$$

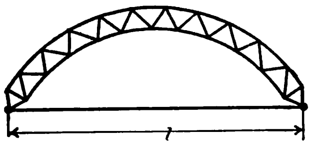

Fig. 56

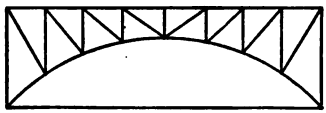

Fig. 57

And in the case of spandrel-braced arch (Fig. 58), from Art. 66, we obtain similarly,

$$H = \frac{\sum_{0}^{na}\left\{\frac{pa^2(s_0-s)}{s^2A_u}+\frac{pas_0b}{h^2A_l}+\frac{es_0s}{(e-pa)^2A_v}+\frac{es_0d}{f^2A_d}\right\}+\frac{(s_0-s_1)s_0}{aA_{ab}}+\sum_{na}^{\frac{l}{2}}\left\{\frac{na^2(s_0-s)}{s^2A_u}+\frac{nas_0b}{h^2A_l}+\frac{nas_0s}{(e-pa)^2A_v}+\frac{nas_0d}{f^2A_d}\right\}}{2\sum_{0}^{\frac{l}{2}}\left\{\frac{(s_0-s)^2a}{s^2A_u}+\frac{s_0^2b}{h^2A_l}+\frac{s_0^2s}{(e-pa)^2A_v}+\frac{s_0^2d}{f^2A_d}\right\}+2\frac{(s_0-s_1)^2s_0}{a^2A_{ab}}+\frac{l}{A_t}} W \quad (132)$$

The effect of introducing the tie, on the amount of H, becomes conspicuous with diminished rise of the arch and increased moment of inertia, as will be seen in the case of flat parabolic arch, for which we have, from Art. 58, the following approximate equation for H due to one W:

$$H = \frac{5a(l-a)(l^2+al-a^2)}{l^3\left(8h + \frac{15I}{A_th}\right)} W.$$

EXAMPLE. — Taking the circular arch of Fig. 50, given as an example in Art. 70, and adding to the same the horizontal tie with a cross-section of 50 square inches (Fig. 58), to calculate the stress in the tie, due to a uniform load of 20 tons per panel.

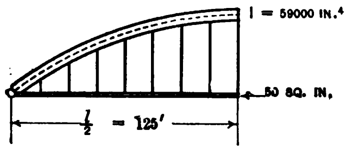

Fig. 58

Transforming Eq. (131) according to Art. 51, we have for the circular arch, by neglecting axial stress,

$$H = \frac{(r-h)\{2e + \phi_a(l - 2a) - \phi_0 l\} + a(l-a)}{2\phi_0\{r^2 + 2(r-h)^2\} - 3l(r-h) + \dfrac{2Il}{A_t r}} W.$$

Referring to the data of the previous case, the denominator in this case equals

$$2748 + \frac{2 \times 59{,}000 \times 250}{50 \times 200 \times 144} = 2768;$$

and since we have for the numerator as before,

$$20 \times 30{,}578 = 611{,}560,$$

we get

$$H = \frac{611{,}560}{2768} = 220.94 \text{ tons},$$

which is the stress in the tie.

CHAPTER V

ARCHES WITHOUT HINGES

73. IN this class of arches, since ends are fixed, there will be moments produced at these points whenever the resultant forces do not pass through them. Here we have, then, two more statically-indeterminate forces than in arches with two hinges.

Fig. 59 shows a symmetrical arch-rib loaded vertically with W.

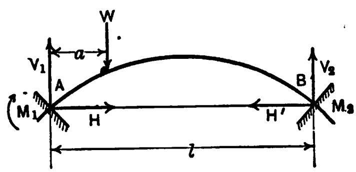

Fig. 59

Let M_1 and M_2 represent moments at A and B respectively. For all other designations, retaining those of the preceding chapter, we have, since the loading is vertical:

$$H - H' = 0,$$
$$V_1 + V_2 - W = 0.$$

For moment at any point distant x from A, we get,

$$m = M_1 + V_1 x - Hy, \text{ for } x < a,$$
$$m = M_1 + V_1 x - Hy - W(x - a), \text{ for } x > a;$$

for vertical shear,

$$V = V_1 \quad \text{for } x < a,$$
$$V = V_1 - W \text{ for } x > a;$$

and for the normal stress in the rib at x (Art. 43),

$$N = -(V \sin \phi + H \cos \phi).$$

Since the internal work in the arch-rib is generally (Art. 44)

$$\omega=\int_0^{l'}\frac{m^2dc}{2EI}+\int_0^{l'}\frac{N^2dc}{2AE},$$

substituting in this, the values of m and N, we get,

$$\omega=\int_0^{a'}\frac{(M_1+V_1x-Hy)^2dc}{2EI}+\int_{a'}^{l'}\frac{\{M_1+V_1x-Hy-W(x-a)\}^2dc}{2EI}$$
$$+\int_0^{a'}\frac{(V_1\sin\phi+H\cos\phi)^2dc}{2EA}+\int_{a'}^{l'}\frac{\{(V_1-W)\sin\phi+H\cos\phi\}^2dc}{2EA}.$$

Since H, M_1 and V_1 must successively make ω a minimum, we get for

$$\frac{d\omega}{dH}=0,\qquad \frac{d\omega}{dM_1}=0,\qquad \frac{d\omega}{dV_1}=0,$$

the following equations:

$$M_1\int_0^{l'}\frac{ydc}{I}+V_1\left(\int_0^{l'}\frac{xydc}{I}-\int_0^{l}\frac{\sin\phi dx}{A}\right)-H\left(\int_0^{l'}\frac{y^2dc}{I}\right.$$
$$\left.+\int_0^{l}\frac{\cos\phi dx}{A}\right)-W\left(\int_{a'}^{l'}\frac{(x-a)\,ydc}{I}-\int_a^{l}\frac{\sin\phi dx}{A}\right)=0. \quad (133)$$

$$M_1\int_0\frac{dc}{I}+V_1\int_0^{l'}\frac{xdc}{I}-H\int_0^{l'}\frac{ydc}{I}-W\int_{a'}^{l'}\frac{(x-a)dc}{I}=0. \quad (134)$$

$$M_1\int_0^{l'}\frac{xdc}{I}+V_1\left(\int_0^{l'}\frac{x^2dc}{I}+\int_0^{l}\frac{\sin\phi dy}{A}\right)-H\left(\int_0^{l'}\frac{xydc}{I}\right.$$
$$\left.-\int_0^{l}\frac{\cos\phi dy}{A}\right)-W\left(\int_{a'}^{l'}\frac{x\,(x-a)\,dc}{I}+\int_a^{l}\frac{\sin\phi dy}{A}\right)=0. \quad (135)$$

These equations will give all the required values of M_1, H, and V_1, as soon as the form of the arch and mode of loading are known.

As to M_2 and V_2 we have,

$$M_2=M_1+V_1l-W(l-a),$$
$$V_2=W-V_1.$$

74. For obtaining expressions for H and M_1 only, it will be more convenient to assume two symmetrical loads

(Fig. 60), as done in the case of two-hinged arches. Denoting the horizontal and vertical reactions and moments at A due to $2\,W$ by H', V', and M', we get,

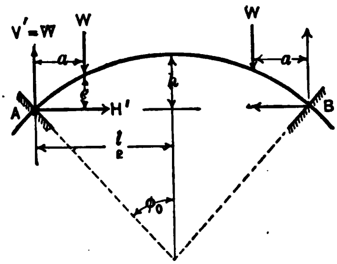

Fig. 60

$$H' = 2\,H,$$
$$M' = M_1 + M_2,$$
$$V' = V_1 + V_2 = W,$$

in which H, M_1, M_2, V_1, and V_2, denote the reactions and moments due to one W, as before.

Referring to the figure, we have for the total internal work in the arch,

$$\omega = 2\int_0^{a'} \frac{(M' + Wx - H'y)^2\,dc}{2\,EI} + 2\int_{a'}^{\frac{l'}{2}} \frac{(M' + Wa - H'y)^2\,dc}{2\,EI}$$
$$+ 2\int_0^{a'} \frac{(W \sin\phi + H'\cos\phi)^2\,dc}{2\,EA} + 2\int_{a'}^{\frac{l'}{2}} \frac{(H'\cos\phi)^2\,dc}{2\,EA},$$

whence for

$$\frac{d\omega}{dH'} = 0, \text{ and } \frac{d\omega}{dM'} = 0,$$

we get,

$$M'\int_0^{\frac{l'}{2}} \frac{y\,dc}{I} - H'\left(\int_0^{\frac{l'}{2}} \frac{y^2dc}{I} + \int_0^{\frac{l'}{2}} \frac{\cos^2\phi\,dc}{A}\right)$$
$$+ W\left(\int_0^{a'} \frac{xy\,dc}{I} + \int_{a'}^{\frac{l'}{2}} \frac{ay\,dc}{I} - \int_0^{a'} \frac{\sin\phi\cos\phi\,dc}{A}\right) = 0 \quad . \quad . \quad (136)$$

$$M'\int_0^{\frac{l'}{2}} \frac{dc}{I} - H'\int_0^{\frac{l'}{2}} \frac{y\,dc}{I} + W\left(\int_0^{a'} \frac{x\,dc}{I} + \int_{a'}^{\frac{l'}{2}} \frac{a\,dc}{I}\right) = 0 \quad . \quad . \quad (137)$$

75. Temperature Stresses. — A uniform temperature change of t degrees would produce a change of $t\theta l$ in the span length of the arch — θ denoting the coefficient of expansion — were the end of the latter free to slide. Designating by

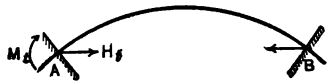

Fig. 61

H_t and M_t (Fig. 61)

the horizontal reaction and moment at A due to a temperature change — positive for rise — we get,

$$\omega = \int_0^{l'} \frac{(M_t - H_t y)^2\,dc}{2\,IE} + \int_0^{l'} \frac{(H_t \cos\phi)^2\,dc}{2\,AE}\,.$$

Since, according to the theorems of Castigliano (Art. 6),

$$\frac{d\omega}{dH_t} = t\theta l, \qquad \frac{d\omega}{dM_t} = 0,$$

we get

$$\int_0^{l'} \frac{-M_t y dc + H_t y^2 dc}{IE} + \int_0^{l'} \frac{H_t \cos^2\phi\,dc}{AE} = t\theta l,$$

$$\int_0^{l'} \frac{(M_t - H_t y)\,dc}{I} = 0,$$

from which

$$H_t = \frac{t\theta l E}{\displaystyle\int_0^{l'} \frac{y^2 dc}{I} + \int_0^{l} \frac{\cos\phi\,dx}{A} - \frac{\left(\displaystyle\int_0^{l'} \frac{ydc}{I}\right)^2}{\displaystyle\int_0^{l'} \frac{dc}{I}}}\,. \quad . \quad (138)$$

$$M_t = \frac{\displaystyle\int_0^{l'} \frac{ydc}{I}}{\displaystyle\int_0^{l'} \frac{dc}{I}}\,H_t\,. \quad . \quad . \quad . \quad . \quad . \quad (139)$$

76. Stresses Due to Displacements of Supports. — The supports may sometimes yield to a certain extent, producing changes in their relative heights as well as the central angle and the span length of the arch.

Representing by

$$M_\Delta,\ V_\Delta, \text{ and } H_\Delta,$$

the moment and the vertical and horizontal reactions at the left end of the arch, caused by such displacements, we get for the internal work in the arch,

$$\omega = \int_0^{l'} \frac{(M_\Delta + V_\Delta x - H_\Delta y)^2\, dc}{2\,EI} + \int_0^{l'} \frac{(H_\Delta \cos\phi + V_\Delta \sin\phi)^2\, dc}{2\,EA}.$$

77. Let Δy = change in relative heights of supports — measured at the left support in the direction of the force, i.e., negative downward. Then, since the force acting through Δy is V_Δ only, according to the theorem of Castigliano, we have,

$$\frac{d\omega}{dV_\Delta} = \Delta y,$$

$$\frac{d\omega}{dH_\Delta} = 0,$$

$$\frac{d\omega}{dM_\Delta} = 0;$$

whence we get,

$$M_\Delta \int_0^{l'} \frac{x\,dc}{I} + V_\Delta \left(\int_0^{l'} \frac{x^2 dc}{I} + \int_0^{l'} \frac{\sin^2\phi\, dc}{A} \right) - H_\Delta \left(\int_0^{l'} \frac{xy\,dc}{I} - \int_0^{l'} \frac{\cos\phi \sin\phi\, dc}{A} \right) = E\Delta y. \quad (140a)$$

$$-M_\Delta \int_0^{l'} \frac{y\,dc}{I} - V_\Delta \left(\int_0^{l'} \frac{xy\,dc}{I} - \int_0^{l'} \frac{\sin\phi\ \cos\phi\, dc}{A} \right) + H_\Delta \left(\int_0^{l'} \frac{y^2 dc}{I} + \int_0^{l'} \frac{\cos^2\phi\, dc}{A} \right) = 0 \quad . \quad . \quad (140b)$$

$$M_\Delta \int_0^{l'} \frac{dc}{I} + V_\Delta \int_0^{l'} \frac{x\,dc}{I} - H_\Delta \int_0^{l'} \frac{y\,dc}{I} = 0 \quad . \quad . \quad . \quad . \quad (140c)$$

78. Next, let $\Delta\phi$ = total change of the central angle of the arch — measured at the left support in the sense of the moment, i.e., positive for the decrease of the central angle. Then, for similar reasons as before, we have,

$$\frac{d\omega}{dM_\Delta} = \Delta\phi,$$

$$\frac{d\omega}{dH_\Delta} = 0,$$

$$\frac{d\omega}{dV_\Delta} = 0;$$

whence,

$$M_\Delta \int_0^{l'} \frac{dc}{I} + V_\Delta \int_0^{l'} \frac{xdc}{I} - H_\Delta \int_0^{l'} \frac{ydc}{I} = E\Delta\phi \quad . \quad . \quad . \quad (141a)$$

$$-M_\Delta \int_0^{l'} \frac{ydc}{I} - V_\Delta \left(\int_0^{l'} \frac{xydc}{I} - \int_0^{l'} \frac{\sin\phi\cos\phi\,dc}{A} \right)$$
$$+ H_\Delta \left(\int_0^{l'} \frac{y^2dc}{I} + \int_0^{l'} \frac{\cos^2\phi\,dc}{A} \right) = 0 \quad . \quad . \quad . \quad (141b)$$

$$M_\Delta \int_0^{l'} \frac{xdc}{I} + V_\Delta \left(\int_0^{l'} \frac{x^2dc}{I} + \int_0^{l'} \frac{\sin^2\phi\,dc}{A} \right)$$
$$- H_\Delta \left(\int_0^{l'} \frac{xydc}{I} - \int_0^{l'} \frac{\sin\phi\cos\phi\,dc}{A} \right) = 0 \quad . \quad (141c)$$

79. Finally, let Δl = total change in span length — measured at the left support in the direction of the force, i.e., positive for the decrease of the span length.

Here we have,

$$\frac{d\omega}{dH_\Delta} = \Delta l,$$

$$\frac{d\omega}{dV_\Delta} = 0,$$

$$\frac{d\omega}{dM_\Delta} = 0;$$

whence,

$$- M_\Delta \int_0^{l'} \frac{ydc}{I} - V_\Delta \left(\int_0^{l'} \frac{xydc}{I} - \int_0^{l'} \frac{\sin\phi\cos\phi\,dc}{A} \right)$$
$$+ H_\Delta \left(\int_0^{l'} \frac{y^2dc}{I} + \int_0^{l'} \frac{\cos^2\phi\,dc}{A} \right) = E\Delta l \,. \quad . \quad (142a)$$

$$M_\Delta \int_0^{l'} \frac{xdc}{I} + V_\Delta \left(\int_0^{l'} \frac{x^2dc}{I} + \int_0^{l'} \frac{\sin^2 \phi \, dc}{A} \right)$$
$$- H_\Delta \left(\int_0^{l'} \frac{xydc}{I} - \int_0^{l'} \frac{\sin \phi \cos \phi \, dc}{A} \right) = 0 \quad . \quad (142b)$$
$$M_\Delta \int_0^{l'} \frac{dc}{I} + V_\Delta \int_0^{l'} \frac{xdc}{I} - H_\Delta \int_0^{l'} \frac{ydc}{I} = 0 \quad . \quad . \quad . \quad . \quad . \quad (142c)$$

PARABOLIC ARCH WITHOUT HINGES

80. Assuming, as in the case of two-hinged arches (Art. 48), the cross-section of the rib to so vary from the crown toward each end that at any section

$$I = I_0 \sec \phi,$$
$$A = A_0 \sec \phi,$$

(I_0 and A_0 denoting the moment of inertia and cross-section of the rib at the crown), and introducing these together with the equation of parabola

$$y = \frac{4h}{l^2} x (l - x)$$

in Eqs. (136) and (137), and integrating

$$\int_0^{\frac{l'}{2}} \frac{ydc}{I} = \frac{hl}{3I_0},$$

$$\int_0^{\frac{l'}{2}} \frac{y^2dc}{I} = \frac{4h^2l}{15I_0},$$

$$\int_0^{\frac{l'}{2}} \frac{\cos^2 \phi \, dc}{A} = \frac{l^2\phi_0}{8hA_0},$$

$$\int_0^{a'} \frac{xydc}{I} = \frac{ha^3(4l - 3a)}{3l^2I_0},$$

$$\int_{a'}^{\frac{l'}{2}} \frac{aydc}{I} = \frac{ah(l^3 - 6a^2l + 4a^3)}{3l^2I_0},$$

$$\int_0^{a'} \frac{\sin \phi \cos \phi \, dc}{A} = \frac{l^2e}{(l^2 + 16h^2)A_0}, \text{ nearly,*}$$

* Howe, Treatise on Arches.

$$\int_0^{\frac{l'}{2}} \frac{dc}{I} = \frac{l}{2\,I_0},$$

$$\int_0^{a'} \frac{x\,dc}{I} = \frac{a^2}{2\,I_0},$$

$$\int_{a'}^{\frac{l'}{2}} \frac{a\,dc}{I} = \frac{a\,(l - 2\,a)}{2\,I_0},$$

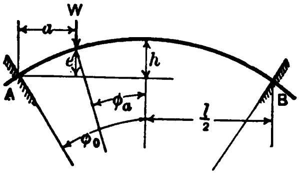

Fig. 62

and putting, as before,

$$\frac{I_0}{A_0} = i^2,$$

we get,

$$M'\frac{hl}{3} - H'\left(\frac{4\,h^2 l}{15} + \frac{l^2 i^2 \phi_0}{8\,h}\right) + W\left\{\frac{ah(l^3 - 2a^2 l + a^3)}{3\,l^2} - \frac{l^2 i^2 e}{l^2 + 16\,h^2}\right\} = 0.$$

$$M'\frac{l}{2} - H'\frac{hl}{3} + W\frac{a\,(l-a)}{2} = 0.$$

Eliminating M', and remembering that

$$e = \frac{4\,h}{l^2}\,a\,(l-a),$$

we obtain,

$$H'\left(\frac{h^2 l}{45} + \frac{l^2 i^2 \phi_0}{16\,h}\right) = W\left\{\frac{a^2 h\,(l-a)^2}{6\,l^2} - \frac{2\,ah\,(l-a)\,i^2}{l^2 + 16\,h^2}\right\}.$$

Consequently, for *one load W* (Fig. 62), we get,

$$H = \tfrac{1}{2}H' = \frac{60\,h^2}{16\,h^3 l + 45\,l^2 i^2 \phi_0}\left\{\frac{a^2\,(l-a)^2}{l^2} - \frac{12\,a\,(l-a)\,i^2}{l^2 + 16\,h^2}\right\}W. \quad (143)$$

Similarly, by carrying out the integrations in Eqs. (134) and (135), we get,

$$M_1 l + V_1\frac{l^2}{2} - H\frac{2}{3}hl - W\frac{(l-a)^2}{2} = 0,$$

$$M_1\frac{l^2}{2} + V_1\left\{\frac{l^3}{3} + \frac{i^2\,(4\,hl - l^2\,\phi_0)}{4\,h}\right\} - H\frac{hl^2}{3}$$

$$- W\left[\frac{(l-a)^2\,(2\,l+a)}{6} + \frac{i^2\{8\,h\,(l-a) - l^2\,(\phi_a + \phi_0)\}}{8\,h}\right] = 0.$$

Eliminating V_1 and M_1 successively, and putting

$$\frac{4hl - l^2\phi_0}{4h} = n,$$

$$\frac{8h(l-a) - l^2(\phi_a + \phi_0)}{8h} = m,$$

$$M_1 = \frac{1}{l^4 + 12lni^2}\left[H\left(\frac{2hl^4}{3} + 8\,hlni^2\right)\right.$$

$$\left. - W\{(al^2 - 6ni^2)(l-a)^2 + 6\,l^2mi^2\}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (144)$$

$$V_1 = \frac{1}{l^3 + 12\,ni^2}\{(l-a)^2(l + 2a) + 12\,mi^2\}W \quad . \quad . \quad . \quad . \quad (145)$$

ϕ_a and ϕ_0 denoting the inclination of tangents at a and A respectively.

Neglecting the effect of axial stress, — since the terms containing i^2 ought then to disappear, — we get,

$$H = \frac{15\,a^2(l-a)^2}{4\,hl^3}W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (146)$$

$$M_1 = \frac{(l-a)^2(5\,a^2 - 2\,al)}{2\,l^3}W \quad . \quad . \quad . \quad . \quad (147)$$

$$V_1 = \frac{(l-a)^2(l + 2\,a)}{l^3}W \quad . \quad . \quad . \quad . \quad . \quad . \quad (148)$$

81. Temperature Stress. — For a uniform temperature change of t, introducing in Eqs. (138) and (139) the equation of parabola and the expressions for I and A already given, and integrating the terms severally, we obtain,

$$H_t = \frac{t\theta EI_0}{\frac{4\,h^2}{45} + \frac{i^2 l\phi_0}{4\,h}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (149)$$

$$M_t = \tfrac{2}{3}hH_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150)$$

Neglecting the axial stress, these equations become,

$$H_t = \frac{45\, t\theta EI_0}{4\, h^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

$$M_t = \frac{15\, t\theta EI_0}{2\, h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

82. Displacement Stresses. — For a change of Δy in relative heights of supports (Art. 77), we get, by carrying out the integrations in Eqs. (140) and combining them,

$$V_\Delta = \frac{EI_0 \Delta y}{l\left\{\frac{l^2}{12} + \frac{(4\, h - l\, \phi_0)\, i^2}{4\, h}\right\}} \quad . \quad . \quad (153)$$

$$M_\Delta = -\, V_\Delta \frac{l}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (154)$$

$$H_\Delta = 0.$$

For a change of $\Delta\phi$ in the central angle (Art. 78), similarly we get from Eq. (141) the following equations:

$$H_\Delta = \frac{2\, hEI_0 \Delta\, \phi}{l\left(\frac{4\, h^2}{15} + \frac{3\, l\, \phi_0 i^2}{4\, h}\right)} \quad . \quad . \quad . \quad . \quad (155)$$

$$V_\Delta = -\, \frac{2\, EI_0\, \Delta\phi}{\frac{l^2}{3} + i^2\left(\frac{4\, h - l\, \phi_0}{h}\right)} \quad . \quad . \quad . \quad (156)$$

$$M_\Delta = -\, V_\Delta \frac{l}{2} + H_\Delta \frac{2\, h}{3} + \frac{EI_0\, \Delta\phi}{l} \quad . \quad (157)$$

For a change of Δl in span length (Art. 79), we get from Eq. (142),

$$H_\Delta = \frac{EI_0\, \Delta\, l}{l\left(\frac{4\, h^2}{45} + \frac{i^2\, l\, \phi_0}{4\, h}\right)} \quad . \quad . \quad . \quad . \quad (158)$$

$$M_\Delta = H_\Delta \frac{2}{3} h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

$$V_\Delta = 0.$$

CIRCULAR ARCH WITHOUT HINGES.

(*Uniform Cross-Section*)

83. Referring to Fig. 63, we have for circular arc with origin of coördinates at A the following relations:

$$x = r(\sin\phi_0 - \sin\phi),$$
$$dx = -r\cos\phi\, d\phi,$$
$$y = r(\cos\phi - \cos\phi_0),$$
$$dy = -r\sin\phi\, d\phi,$$
$$dc = -r d\phi,$$
$$a = r(\sin\phi_0 - \sin\phi_a),$$

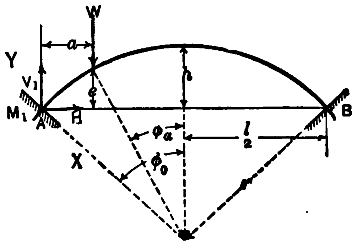

Fig. 63

ϕ denoting the inclination of the tangent to the horizontal at any point xy.

Introducing these in Eqs. (133), (134), and (135), and integrating the terms severally,

(*a*)

$$\int_0^{l'} y dc = 2r^2(\sin\phi_0 - \phi_0\cos\phi_0).$$

$$\int_0^{l'} xy dc = 2r^3\sin\phi_0(\sin\phi_0 - \phi_0\cos\phi_0).$$

$$\int_0^{l}\sin\phi\, dx = 0.$$

$$\int_0^{l'} y^2 dc = r^3\{\phi_0(1 + 2\cos^2\phi_0) - 3\sin\phi_0\cos\phi_0\}.$$

$$\int_0^{l}\cos\phi\, dx = r(\phi_0 + \sin\phi_0\cos\phi_0).$$

$$\int_{a'}^{l'}(x - a) y dc = r^3\{\tfrac{1}{2}(\sin\phi_0 + \sin\phi_a)^2 - (\phi_0 + \phi_a)\sin\phi_a\cos\phi_0$$
$$+ \cos\phi_0(\cos\phi_0 - \cos\phi_a)\}.$$

$$\int_a^{l}\sin\phi\, dx = \frac{r}{2}(\sin^2\phi_a - \sin^2\phi_0).$$

(*b*)

$$\int_0^{l'} dc = 2\,r\,\phi_0.$$

$$\int_0^{l'} xdc = 2\,r^2\,\phi_0 \sin\phi_0.$$

$$\int_0^{l'} ydc = 2\,r^2\,(\sin\phi_0 - \phi_0\cos\phi_0).$$

$$\int_{a'}^{l'} (x-a)\,dc = r^2\,\{(\phi_0+\phi_a)\sin\phi_a + \cos\phi_a - \cos\phi_0\}.$$

(*c*)

$$\int_0^{l'} xdc = 2\,r^2\,\phi_0\sin\phi_0.$$

$$\int_0^{l'} x^2dc = r^3\,\{\phi_0\,(1 + 2\sin^2\phi_0) - \cos\phi_0\sin\phi_0\}.$$

$$\int_0^{l} \sin\phi\,dy = r\,(\phi_0 - \cos\phi_0\sin\phi_0).$$

$$\int_0^{l'} xydc = 2\,r^3\sin\phi_0\,(\sin\phi_0 - \phi_0\cos\phi_0).$$

$$\int_0^{l} \cos\phi\,dy = 0.$$

$$\int_{a'}^{l'} (x-a)\,xdc = r^3\,\{(\phi_0+\phi_a)\,(\tfrac{1}{2} + \sin\phi_0\sin\phi_a) + (\sin\phi_0+\sin\phi_a)(\cos\phi_a - \cos\phi_0) - \tfrac{1}{2}(\cos\phi_0\sin\phi_0 + \sin\phi_a\cos\phi_a)\}.$$

$$\int_a^{l} \sin\phi\,dy = \frac{r}{2}\,\{(\phi_0+\phi_a) - \cos\phi_0\sin\phi_0 - \cos\phi_a\sin\phi_a\}.$$

And eliminating M_1, V_1, and H successively from the three equations, we obtain,

$$H = \frac{\left\{\begin{array}{c}\sin\phi_0(\cos\phi_a - \cos\phi_0) + \sin\phi_a(\phi_a\sin\phi_0 - \phi_0\sin\phi_a) \\ +\dfrac{\phi_0}{2}\left(1+\dfrac{i^2}{r^2}\right)(\sin^2\phi_a - \sin^2\phi_0)\end{array}\right\}}{\phi_0\left(1+\dfrac{i^2}{r^2}\right)(\phi_0 + \cos\phi_0\sin\phi_0) - 2\sin^2\phi_0}\,W. \quad (160a)$$

or, since

$$\sin \phi_0 = \frac{l}{2r}, \qquad \cos \phi_0 = \frac{r-h}{r},$$

$$\sin \phi_a = \frac{l-2a}{2r}, \qquad \cos \phi_a = \frac{r-h+e}{r},$$

$$H = \frac{le + \left(\frac{l}{2} - a\right)\{\phi_a l - \phi_0 (l - 2a)\} - \phi_0 \left(1 + \frac{i^2}{r^2}\right)(l-a)a}{\phi_0 \left(1 + \frac{i^2}{r^2}\right)\{2r^2 \phi_0 + l(r-h)\} - l^2} W. \quad (160b)$$

$$M_1 = \frac{r}{\phi_0}(\sin \phi_0 - \phi_0 \cos \phi_0) H + \frac{r}{2\left(1 + \frac{i^2}{r^2}\right)(\phi_0 - \cos \phi_0 \sin \phi_0)} \times$$

$$\Big[\sin \phi_a (\sin \phi_0 \cos \phi_0 - \sin \phi_0 \cos \phi_a + \phi_0) - \phi_a \sin \phi_0$$

$$- \frac{1}{\phi_0}\left(1 + \frac{i^2}{r^2}\right)(\cos \phi_0 \sin \phi_0 - \phi_0)(\phi_a \sin \phi_a - \phi_0 \sin \phi_0$$

$$+ \cos \phi_a - \cos \phi_0) + \frac{i^2}{r^2}\{\sin \phi_a (\cos \phi_a \sin \phi_0$$

$$- \cos \phi_0 \sin \phi_0 + \phi_0) - \phi_a \sin \phi_0\}\Big] W \; . \; . \; . \; . \; . \; . \quad (161a)$$

or

$$M_1 = \left(\frac{l}{2\phi_0} - r + h\right) H + \frac{1}{4\phi_0 \left(1 + \frac{i^2}{r^2}\right)\{2r^2\phi_0 - l(r-h)\}} \times$$

$$\Big[(l - 2a)(2r^2\phi_0 - le)\phi_0 - 2\phi_0 \phi_a r^2 l$$

$$- \left(1 + \frac{i^2}{r^2}\right)\{2r^2\phi_0 - l(r-h)\}\{2\phi_a a + l(\phi_0 - \phi_a) - 2e\}$$

$$+ \frac{i^2}{r^2}\phi_0 \{(l-2a)(2r^2\phi_0 + le) - 2r^2\phi_a l\}\Big] W \; . \; . \; . \quad (161b)$$

$$V_1 = \left\{\frac{1}{2} + \frac{\left(1 + \frac{i^2}{r^2}\right)(\phi_a - \sin \phi_a \cos \phi_a) + 2\sin \phi_a (\cos \phi_a - \cos \phi_0)}{2\left(1 + \frac{i^2}{r^2}\right)(\phi_0 - \cos \phi_0 \sin \phi_0)}\right\} W. \quad (162a)$$

or

$$V_1=\left[\frac{1}{2}+\frac{\left(1+\frac{i^2}{r^2}\right)\{2\phi_a r^2-(l-2a)(r-h+e)\}+2e(l-2a)}{2\left(1+\frac{i^2}{r^2}\right)\{2r^2\phi_0-l(r-h)\}}\right]W. \quad (162b)$$

Neglecting the effect of axial stress, we get,

$$H=\frac{\left\{\begin{array}{l}\sin\phi_0(\cos\phi_a-\cos\phi_0)+\sin\phi_a(\phi_a\sin\phi_0-\phi_0\sin\phi_a)\\ \quad+\frac{\phi_0}{2}(\sin^2\phi_a-\sin^2\phi_0)\end{array}\right\}}{\phi_0(\phi_0+\cos\phi_0\sin\phi_0)-2\sin^2\phi_0}W. \quad (163a)$$

or

$$H=\frac{le+\left(\frac{l}{2}-a\right)\{\phi_a l-\phi_0(l-2a)\}-\phi_0 a(l-a)}{\phi_0\{2r^2\phi_0+l(r-h)\}-l^2}W. \quad (163b)$$

$$M_1=\frac{r}{\phi_0}(\sin\phi_0-\phi_0\cos\phi_0)H+\frac{r}{2(\phi_0-\cos\phi_0\sin\phi_0)}\times$$

$$\Big[\sin\phi_a(\sin\phi_0\cos\phi_0-\sin\phi_0\cos\phi_a+\phi_0)-\phi_a\sin\phi_0$$

$$-\frac{1}{\phi_0}(\cos\phi_0\sin\phi_0-\phi_0)(\phi_a\sin\phi_a-\phi_0\sin\phi_0+\cos\phi_a-\cos\phi_0)\Big]W. \quad (164a)$$

or

$$M_1=\left(\frac{l}{2\phi_0}-r+h\right)H+\frac{1}{4\phi_0\{2r^2\phi_0-l(r-h)\}}\times$$

$$\Big[(l-2a)(2r^2\phi_0-le)\phi_0-2\phi_0\phi_a r^2 l$$

$$-\{2r^2\phi_0-l(r-h)\}\ \{2\phi_a a+l(\phi_0-\phi_a)-2e\}\Big]W. \quad . \quad (164b)$$

$$V_1=\left\{\frac{1}{2}+\frac{(\phi_a-\sin\phi_a\cos\phi_a)+2\sin\phi_a(\cos\phi_a-\cos\phi_0)}{2(\phi_0-\cos\phi_0\sin\phi_0)}\right\}W. \quad (165a)$$

or

$$V_1=\left[\frac{1}{2}+\frac{\{2\phi_a r^2-(l-2a)(r-h+e)\}+2e(l-2a)}{2\{2r^2\phi_0-l(r-h)\}}\right]W. \quad (165b)$$

84. Temperature Stresses. — For a uniform temperature change of t — positive for rise — we get, by carrying out the integrations as before in Eqs. (138) and (139),

$$H_t = \frac{t\theta l E I \phi_0}{r^3\left[\phi_0(\phi_0 + \sin\phi_0\cos\phi_0)\left(1+\frac{i^2}{r^2}\right) - 2\sin^2\phi_0\right]} \quad (166a)$$

$$= \frac{2\,t\theta l E I \phi_0}{r\left[\phi_0\{2\,\phi_0 r^2 + l(r-h)\}\left(1+\frac{i^2}{r^2}\right) - l^2\right]} \quad (166b)$$

$$M_t = H\,\frac{(\sin\phi_0 - \phi_0\cos\phi_0)\,r}{\phi_0} \quad (167a)$$

$$= H\,\frac{l - 2\,\phi_0(r-h)}{2\,\phi_0} \quad (167b)$$

Neglecting axial stress, we get,

$$H_t = \frac{t\theta l E I \phi_0}{r^3[\phi_0(\phi_0 + \sin\phi_0\cos\phi_0) - 2\sin^2\phi_0]} \quad (168a)$$

$$= \frac{2\,t\theta l E I \phi_0}{r[\phi_0\{2\phi_0 r^2 + l(r-h)\} - l^2]} \quad (168b)$$

85. Displacement Stresses. — For a change of Δy (Art. 77), we get, by carrying out the integrations in Eq. (140),

$$V_\Delta = \frac{EI\,\Delta y}{r^3\left(1+\frac{i^2}{r^2}\right)(\phi_0 - \cos\phi_0\sin\phi_0)} \quad (169a)$$

$$= \frac{2\,EI\,\Delta y}{r\left(1+\frac{i^2}{r^2}\right)\{2\,r^2\phi_0 - l(r-h)\}} \quad (169b)$$

$$M_\Delta = -V_\Delta r\sin\phi_0 = -V_\Delta\frac{l}{2} \quad (170)$$

$$H_\Delta = 0.$$

For a change of $\Delta\phi$ in the central angle (Art. 78), we get from Eq. (141),

$$V_\Delta = -\frac{EI\sin\phi_0\,\Delta_0}{r^2\left(1+\frac{i^2}{r^2}\right)(\phi_0 - \cos\phi_0\sin\phi_0)} \quad (171a)$$

$$= -\frac{EI\,l\,\Delta\phi}{r\left(1+\frac{i^2}{r^2}\right)\{2\,r^2\phi_0 - l(r-h)\}} \quad (171b)$$

$$H_\Delta = \frac{EI\,(\sin\phi_0 - \phi_0\cos\phi_0)\,\Delta\phi}{r^2\left\{\phi_0\left(1+\frac{i^2}{r^2}\right)(\phi_0+\sin\phi_0\cos\phi_0 - 2\sin^2\phi_0)\right\}} \quad (172a)$$

$$= \frac{EI\,\{l - 2\,\phi_0(r-h)\}\,\Delta\phi}{r\left[\phi_0\,\{2\,\phi_0 r^2 + l(r-h)\}\left(1+\frac{i^2}{r^2}\right) - l^2\right]} \quad . \quad . \quad . \quad (172b)$$

$$M_\Delta = H_\Delta\frac{r}{\phi_0}(\sin\phi_0 - \phi_0\cos\phi_0) - V_\Delta r\sin\phi_0 + \frac{EI\,\Delta\phi}{2\,\phi_0 r} \quad (173a)$$

$$= H_\Delta\frac{l - 2\,\phi_0(r-h)}{2\,\phi_0} - V_\Delta\frac{l}{2} + \frac{EI\,\Delta\phi}{2\,\phi_0 r} \quad . \quad . \quad . \quad . \quad (173b)$$

For a change of Δl in span length (Art. 79), we obtain from Eq. (142),

$$H_\Delta = \frac{EI\,\phi_0\Delta l}{r^3\left[\phi_0\left(1+\frac{i^2}{r^2}\right)(\phi_0+\sin\phi_0\cos\phi_0) - 2\sin^2\phi_0\right]} \quad (174a)$$

$$= \frac{2\,EI\,\phi_0\Delta l}{r\left[\phi_0\,\{2\,r^2\phi_0 + l(r-h)\}\left(1+\frac{i^2}{r^2}\right) - l^2\right]} \quad . \quad . \quad . \quad (174b)$$

$$M_\Delta = H_\Delta\frac{r\,(\sin\phi_0 - \phi_0\cos\phi_0)}{\phi_0} = H_\Delta\frac{l - 2\,\phi_0(r-h)}{2\,\phi_0} \quad . \quad (175)$$

$$V_\Delta = 0.$$

FLAT ARCH WITHOUT HINGES

86. In arches with comparatively small versed-sines, we may put, as before, without material error,

$$dc = dx,$$

so that

$$\int_0^l \sin\phi\,dx = 0,$$

$$\int_0^l \cos\phi\,dx = l,$$

$$\int \sin\phi\,dy = 0,$$

$$\int_0^l \cos\phi\,dy = 0.$$

Introducing these in Eqs. (133), (134), and (135), and assuming the cross-section of the rib to be uniform throughout, with

$$\frac{I}{A} = i^2,$$

we get,

$$M_1 \int_0^l y dx + V_1 \int_0^l xy dx - H \left(\int_0^l y^2 dx + li^2 \right)$$
$$- W \int_a^l (x - a) y dx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (176a)$$

$$M_1 l + V_1 \frac{l^2}{2} - H \int_0^l y dx - W \frac{(l - a)^2}{2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (176b)$$

$$M_1 \frac{l^2}{2} + V_1 \frac{l^3}{3} - H \int_0^l xy dx - W \frac{(2l + a)(l - a)^2}{6} = 0 \quad . \quad (176c)$$

Combining these equations, we get,

$$V_1 = \frac{(l + 2a)(l - a)^2}{l^3} W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (177)$$

$$H = \frac{\int_a^l (x - a) y dx - \frac{(l - a)^2}{2} \int_0^l y dx}{\left(\int_0^l y dx \right)^2 - l \int_0^l y^2 dx - l^2 i^2} W \quad . \quad . \quad (178)$$

$$M_1 = H \frac{\int_0^l y dx}{l} - W \frac{a(l - a)^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (179)$$

It is to be noted that $\int_0^l y dx$ is the area above the horizontal line joining the ends of the arch and bounded by the axis of the latter.

87. Temperature Stresses. — For a uniform temperature change of t (Art. 75), similarly we get from Eqs. (138) and (139),

$$H_t = \frac{t\theta l^2 EI}{l\int_0^l y^2 dx + l^2 i^2 - \left(\int^l ydx\right)^2} \quad . \quad . \quad . \quad . \quad (180)$$

$$M_t = H_t \frac{\int_0^l ydx}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

88. Displacement Stresses. — Similarly from Eqs. (140), (141), and (142), we get, for Δy (Art. 77),

$$V_\Delta = \frac{12\, EI\, \Delta\, y}{l^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (182)$$

$$M_\Delta = -V_\Delta \frac{l}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (183)$$

for $\Delta\phi$ (Art. 78),

$$H_\Delta = \frac{EI\, \Delta\phi \int_0^l ydx}{l\int_0^l y^2 dx - \left(\int_0^l ydx\right)^2 + l^2 i^2} \quad . \quad . \quad . \quad (184)$$

$$V_\Delta = -\frac{6\, EI\, \Delta\phi}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (185)$$

$$M_\Delta = -V_\Delta \frac{2}{3} l + H_\Delta \frac{\int_0^l ydx}{l} \quad . \quad . \quad . \quad . \quad (186)$$

and for Δl (Art. 79),

$$H_\Delta = \frac{lEI\, \Delta\, l}{l\int_0^l y^2 dx - \left(\int_0^l ydx\right)^2 + l^2 i^2} \quad . \quad . \quad . \quad (187)$$

$$M_\Delta = H_\Delta \frac{\int_0^l ydx}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (188)$$

FLAT PARABOLIC ARCH WITHOUT HINGES

(Uniform Cross-Section.)

89. Introducing in Eqs. (177), (178), and (179) the equation of parabola

$$y = \frac{4h}{l^2} x (l - x),$$

we get for one load W (Fig. 64) the following equations:

$$H = \frac{15 a^2 (l - a)^2}{4 h l^3 \left(1 + \frac{45 i^2}{4 h^2}\right)} W \quad . \quad . \quad . \quad . \quad (189)$$

$$M_1 = -\frac{a (l - a)^2}{2 l^3} \left(2 l - \frac{5 a}{1 + \frac{45 i^2}{4 h^2}}\right) W \quad . \quad . \quad (190)$$

$$V_1 = \frac{(l + 2 a)(l - a)^2}{l^3} W \quad . \quad . \quad . \quad . \quad (191)$$

Neglecting the effect of axial compression,

$$H = \frac{15 a^2 (l - a)^2}{4 h l^3} W \quad . \quad . \quad . \quad . \quad (192)$$

$$M_1 = -\frac{a (l - a)^2 (2 l - 5 a)}{2 l^3} W \quad . \quad . \quad . \quad . \quad (193)$$

$$V_1 = \frac{(l + 2 a)(l - a)^2}{l^3} W \quad . \quad . \quad . \quad . \quad (194)$$

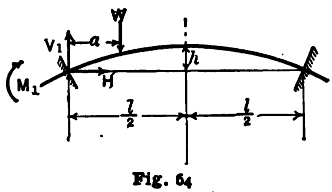

Fig. 64

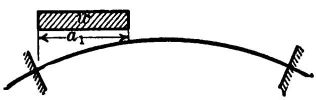

Fig. 65

90. For a uniform load w per unit length of the span (Fig. 65), substituting wda for W, and integrating between the given limits of loading, we get,

$$H = \frac{(10\,l^2 - 15\,a_1 l + 6\,a_1^2)\,a_1^3 w}{8\,hl^3\left(1 + \frac{45\,i^2}{4\,h^2}\right)} \quad . \quad . \quad . \quad . \quad (195)$$

$$M_1 = -\left\{6\,l^2 - 8\,a_1 l + 3\,a_1^2 - \frac{a_1\,(10\,l^2 - 15\,a_1 l + 6\,a_1^2)}{l\left(1 + \frac{45\,i^2}{4\,h}\right)}\right\}\frac{a_1^2 w}{12\,l^2}. \quad (196)$$

$$V_1 = \frac{(2\,l^3 - 2\,a_1^2 l + a_1^3)\,a_1 w}{2\,l^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (197)$$

Neglecting axial stress, we get,

$$H = \frac{(10\,l^2 - 15\,a_1 l + 6\,a_1^2)\,a_1^3 w}{8\,hl^3} \quad . \quad . \quad . \quad . \quad (198)$$

$$M_1 = -\frac{(l - a_1)^3\,a_1^2 w}{2\,l^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (199)$$

$$V_1 = \frac{(2\,l^3 - 2\,a_1^2 l + a_1^3)\,a_1 w}{2\,l^3} \quad . \quad . \quad . \quad . \quad (200)$$

91. Temperature Stresses. — Similarly from Eqs. (180) and (181) we get,

$$H_t = \frac{45\,t\theta EI}{4\,h^2 + 45\,i^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (201)$$

$$M_t = H_t\,\tfrac{2}{3}\,h = \frac{30\,t\theta EIh}{4\,h^2 + 45\,i^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (202)$$

Neglecting axial stress,

$$H_t = \frac{45\,t\theta EI}{4\,h^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (203)$$

$$M_t = \frac{15\,t\theta EI}{2\,h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (204)$$

92. Displacement Stresses. — From Eqs. (182) to (188) we obtain in a similar manner,

for Δy (Art. 77),

$$V_\Delta = \frac{12\,EI\Delta y}{l^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (205)$$

$$M_\Delta = -V_\Delta\,\frac{l}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (206)$$

for $\Delta\phi$ (Art. 78),

$$H_\Delta = \frac{30\,EIh\Delta\phi}{l(4h^2 + 45i^2)} \quad \ldots \ldots \quad (207)$$

$$V_\Delta = -\frac{6\,EI\Delta\phi}{l^2} \quad \ldots \ldots \quad (208)$$

$$M_\Delta = -V_\Delta \tfrac{1}{2}l + H_\Delta \tfrac{2}{3}h \quad \ldots \ldots \quad (209)$$

and for Δl (Art. 79),

$$H_\Delta = \frac{45\,EI\Delta l}{l(4h^2 + 45i^2)} \quad \ldots \ldots \quad (210)$$

$$M_\Delta = H_\Delta \tfrac{2}{3}h \quad \ldots \ldots \quad (211)$$

93. For *flat circular arches* without hinges, the foregoing formulas deduced for parabolic arches may be used without sensible error, for the reason already stated in the case of arches with two hinges (Art. 62).

REACTION LOCUS AND ENVELOPE

94. For showing reactions in an arch without hinges in amount and direction, reaction locus and envelope are required.

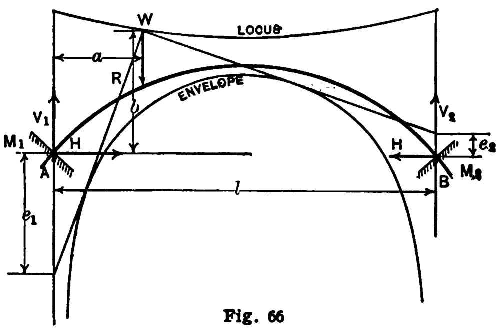

Fig. 66

Since end moments are due to the deviation of reactions from the axis, if we represent by e the vertical distance taken as positive above and negative below the horizontal line connecting the ends of the arch, we have in Fig. 66 at the left end,

$$e_1 = \frac{M_1}{H},$$

and at the right end,

$$e_2 = \frac{M_2}{H}.$$

Since

$$\frac{V_1}{H} = \frac{b - e_1}{a}, \qquad b = e_1 + \frac{V_1 a}{H} = \frac{M_1 + V_1 a}{H};$$

$$\frac{V_2}{H} = \frac{b - e_2}{l - a}, \qquad b = e_2 - \frac{V_2}{H}(l - a) = \frac{M_2 - V_2 (l - a)}{H};$$

which are the equations of the locus.

Next, let $x_1 y_1$ be coördinates — with origin at A — of the point in the line of reaction R, assumed to be one of contact with the envelope. Then we have,

$$y_1 = e_1 + \frac{V_1}{H} x_1 = e_1 + \frac{b - e_1}{a} x_1.$$

In order to find the relation between x_1 and y_1 for variable a, differentiate this equation with respect to a and eliminate a from the same. The equation thus obtained will be that of the envelope.

It is evident that locus and envelope could be drawn by simple plotting of reaction lines for different positions of loads, instead of by deducing their equations.

95. Taking the case of a *flat parabolic arch*, since we have by neglecting the effect of axial stress (Art. 89),

$$V_1 = \frac{(l + 2a)(l - a)^2}{l^3} W,$$

$$H = \frac{15 a^2 (l - a)^2}{4 h l^3} W,$$

$$M_1 = - \frac{a (l - a)^2 (2l - 5a)}{2 l^3} W,$$

we get,

$$e_1 = -\frac{2h(2l - 5a)}{15a},$$

$$b = e_1 + \frac{V_1}{H}a = \frac{6h}{5},$$

showing that the reaction locus is a horizontal line.

Since

$$y_1 = e_1 + \frac{b - e_1}{a}x_1,$$

substituting the values of e_1 and b_1,

$$y_1 = -\frac{2h(2l - 5a)}{15a} + \frac{4h(2a + l)}{15a^2}x_1.$$

Differentiating this with respect to a, we get,

$$a = \frac{2h(l - 2x_1)}{5(2h - 3y_1)}.$$

Substituting this value of a in the preceding equation, and at the same time transferring the origin of coördinates to the centre of the span and in level of the crown of the envelope where

$$y_1 = \frac{2}{3}h \text{ and } x_1 = \frac{l}{2},$$

we get,

$$8hx^2 + 15l^2y + 30lxy = 0,$$

which is the equation of hyperbola.

POSITION OF LOADS FOR MAXIMUM STRESS

96. For finding the mode of loading to produce maximum stress in any part of the arch, reaction locus and envelope may be made use of in a similar manner as explained in the case of two-hinged arches (Art. 68).

In Fig. 67, let the outside lines of the rib represent the positions of centres of gravity of the flanges or chords. Since at any normal section CD of the rib the stress in the upper flange C is equal to the moment with respect to D divided by d, the reaction line passed through D and produced to the locus will indicate the position of load producing no stress in C, and that all loads to the left of O produce compression in C, while those to the right, tension. For the same reason the reaction line drawn through C determines the position for load to produce no stress in D, so that all loads to the left of O' produce tension in D, and those to the right, compression.

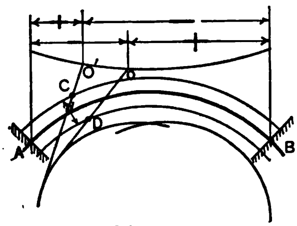

Fig. 67

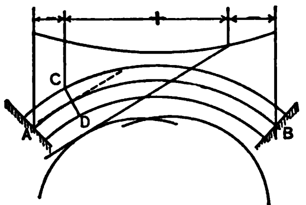

Fig. 68

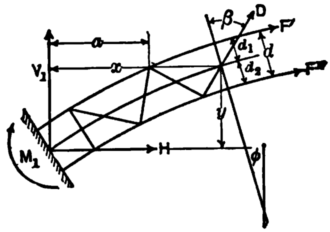

Fig. 69

For similar reasons, the position of load producing no shear at the normal section is given by drawing the reaction line parallel to the tangent to the axis of the arch at the section (Fig. 68) and by erecting a vertical over the section. The loads within these limits evidently produce (Art. 68) positive shear, while those outside the same, the negative.

In case the chords or flanges are not parallel, the reaction line should be passed through the intersection of chord-members of the panel — in which the shear is to be determined — instead of drawing parallel to the tangent, to find the position of load producing no shear.

THE STRESSES IN INDIVIDUAL MEMBERS

97. Referring to Art. 67, it will at once be seen that, in the arch-rib of Fig. 69, we have but to add M_1 to the moment of external forces.

Using the same designations as in Art. 70, we then have,

$$F'' = \frac{1}{d}\left\{ M_1 + V_1\,x - Hy - \sum_0^x W\,(x-a) - (V\sin\phi + H\cos\phi)\,d_1 \right\} \quad \ldots \ldots \quad (212)$$

$$F' = -\frac{1}{d}\left\{ M_1 + V_1\,x - Hy - \sum_0^x W\,(x-a) + (V\sin\phi + H\cos\phi)\,d_2 \right\} \quad \ldots \ldots \quad (213)$$

$$D = -(V\cos\phi - H\sin\phi)\sec\beta \quad \ldots \quad (214)$$

in which $$V = V_1 - \sum_0^x W.$$

For a *non-parallel rib*, the stress in each member is best obtained by taking moment at the intersection of the other two members cut by a section.

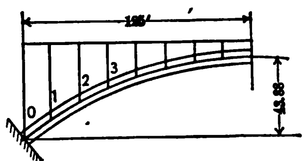

Fig. 70

EXAMPLE. — In a full-webbed circular arch with the same general dimensions as given in the case of two-hinged arch on page 94, to find the maximum stress in the lower flange at 3 (Fig. 70).

Loads and dimensions :

Dead load = 20 tons per panel.
Live load = 10 tons per panel.
l = 250 ft.
r = 200.
$\phi_0 = 38° - 40' - 56'' = .67514.$
Panel length = 15.625 ft.
Cross-section uniform with $\frac{i^2}{r^2} = .00019.$
Effective depth = 6 ft.

Panel.	a (ft.).	e (ft.).	ϕa.	ϕa (circ. meas.).
1	15.63	11.32	33° 09' 10"	.57863
2	31.25	20.54	27 57 11	.48788
3	46.88	27.99	22 59 36	.40114
4	62.50	33.86	18 12 36	.31782
5	78.13	38.31	13 33 17	.23658
6	93.75	41.42	8 59 21	.15689
7	109.38	43.26	4 29 12	.07831
8	125.00	43.88	0	0

The horizontal reaction is given by Eq. (160*b*).

$$H = \frac{le + \left(\frac{l}{2} - a\right)\{\phi_a l - \phi_0 (l - 2a)\} - \phi_0\left(1 + \frac{i^2}{r^2}\right) a (l - a)}{\phi_0\left(1 + \frac{i^2}{r^2}\right)\{2r^2 \phi_0 + l(r - h)\} - l^2} W.$$

Tabulating the terms of the numerator severally, we have,

Panel Pt.	le	$\phi_0 (l - 2a)$	$\phi_a l$	$\left(\frac{l}{2} - a\right)\{\phi_0 (l - 2a) - \phi_a l\}$	$\left(1 + \frac{i^2}{r^2}\right)\phi_0 a (l - a)$	Numerator.
1	2829.46	147.69	144.66	331.21	2472.88	25.37
2	5135.29	126.59	121.97	432.98	4616.03	86.28
3	6996.25	105.49	100.28	406.66	6429.48	160.11
4	8464.64	84.39	79.46	308.46	7918.20	237.98
5	9576.05	63.29	59.14	194.51	9067.20	314.34
6	10354.62	42.20	39.22	92.89	9891.50	370.23
7	10815.54	21.10	19.58	23.76	10386.07	405.70
8	10968.75	0	0	0	10550.93	417.82

and for the denominator,

$$\phi_0\left(1+\frac{i^2}{r^2}\right)\{2\,r^2\,\phi_0+l\,(r-h)\}-l^2=327.15,$$

whence we get the following value of H for $W=1$:

Load at.	1	2	3	4	5	6	7	8
$H=$	.0775	.2637	.4893	.7275	.9605	1.1314	1.2397	1.2768

For M_1 we have Eq. (161*b*).

$$M_1=\left(\frac{l}{2\,\phi_0}-r+h\right)H+\frac{1}{4\,\phi_0\left(1+\frac{i^2}{r^2}\right)\{2\,r^2\phi_0-l\,(r-h)\}}\times$$

$$\Big[\phi_0\,(l-2\,a)\,(2\,r^2\,\phi_0-le)-2\,\phi_0\,\phi_a\,r^2l-\left(1+\frac{i^2}{r^2}\right)\{2r^2\,\phi_0$$

$$-\,l\,(r-h)\}\ \{2\,\phi_a\,a+l\,(\phi_0-\phi_a)-2\,e\}+\frac{i^2}{r^2}\phi_0$$

$$\{(l-2\,a)(2\,r^2\,\phi_0+le)-2\,r^2\,\phi_a\,l\}\Big]W.$$

Tabulating the terms severally,

Panel Pt.	$\left(\frac{l}{2\phi_0}-r+h\right)H$	First Term of Numerator.	Second Term.	Third Term.	Fourth Term.	Numerator.
1	2.25	7,558,673	−7,812,985	− 278,225	11	− 532,526
2	7.65	6,186,976	−6,587,596	− 542,713	17	− 943,316
3	14.20	4,959,501	−5,416,391	− 751,134	19	−1,208,005
4	20.97	3,843,680	−4,291,456	− 918,971	19	−1,366,728
5	27.88	2,811,769	−3,194,392	−1,048,682	16	−1,431,288
6	32.84	1,842,092	−2,118,465	−1,140,741	11	−1,417,102
7	35.98	911,322	−1,057,323	−1,195,747	6	−1,341,743
8	37.06	0	0	−1,214,044	0	−1,214,044

The denominator being

$$4\,\phi_0\left(1+\frac{i^2}{r^2}\right)\{2\,r^2\,\phi_0-l\,(r-h)\}=40{,}459.6,$$

we get the following values of M_1 in ft.-tons; and from $M_2=M_1+V_1l-W\,(l-a)$ the values of M_2:

Load at	1	2	3	4	5	6	7	8
$M_1 =$	−10.91	−15.66	−15.65	−12.81	− 7.50	− 2.19	+ 2.82	+7.05
$M_2 =$	+ 1.02	+ 4.09	+ 6.98	+10.05	+11.38	+11.31	+ 9.95	+7.05

The values of V_1 are obtained from Eq. (162*b*).

$$V_1 = \left[\frac{1}{2} + \frac{\left(1+\frac{i^2}{r^2}\right)\{2\,\phi_a r^2 - (l-2\,a)\,(r-h+e)\} + 2\,e\,(l-2\,a)}{2\left(1+\frac{i^2}{r^2}\right)\{2\,r^2\,\phi_0 - l\,(r-h)\}}\right] W.$$

Substituting the numerical values in all the terms, we get the following values of V_1:

Load at	1	2	3	4	5	6	7	8
V_1	.985	.954	.903	.839	.763	.679	.591	.500

Drawing the reaction locus and envelope (Fig. 71), and passing reaction line through C in the upper flange at 3, we

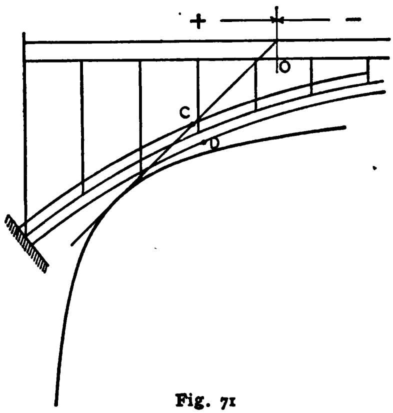

Fig. 71

at once see that all loads to the right of o produce compression in the lower flange at D.

For this position of loads, we have the following end-moment and reactions:

Due to dead load, —

$$H = 2\,(.0775 + .2637 + .4893 + .7275 + .9605 + 1.1314 + 1.2397 + .6384)\,20 = 221.12 \text{ tons.}$$

$$M_1 = (-10.91 - 15.66 - 15.65 - 12.81 - 7.50 - 2.19 + 2.82 + 7.05 + 1.02 + 4.09 + 6.98 + 10.05 + 11.38 + 11.31 + 9.95)\,20 = -2.80 \text{ ft.-tons.}$$

$$V_1 = 7\tfrac{1}{2} \times 20 = 150 \text{ tons.}$$

Due to live load covering the right of o, —

$$H = \{2\,(.9605 + 1.1314 + 1.2397 + .6384) + .0775 + .2637 + .4893 + .7275\}\,10 = 94.93 \text{ tons.}$$

$$M_1 = (-7.50 - 2.19 + 2.82 + 7.05 + 1.02 + 4.09 + 6.98 + 10.05 + 11.38 + 11.31 + 9.95)\,10 = +549.60 \text{ ft.-tons.}$$

$$V_1 = 3\tfrac{1}{2} \times 10 + (.015 + .046 + .097 + .161)\,10 = 38.19 \text{ tons.}$$

From Eq. (212) we then get for flange stress at D,

$$F'' = \tfrac{1}{8}\,[(549.60 - 2.80) + (150 + 38.19)\,46.88 - (221.12 + 94.93)\,27.99 - 20\,(15.63 + 31.25) - \{(188.19 - 40)\,.3906 + 316.05 \times .9205\}\,3] = -243.50 \text{ tons.}$$

CONCLUDING REMARKS ON ARCHES

98. It must be borne in mind that all the formulas that have so far been deduced for arch-ribs apply with correctness only to ribs whose radii of curvature are considerably greater than the depths of the ribs themselves. Were this not the case, the fibre length would differ sensibly with its distance from the centre of curvature, and as a consequence the change in its length will not be proportional merely to the stress acting in the same, but will also depend on its distance from the centre of curvature. Thus in Fig. 72, let

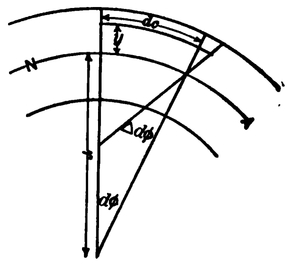
Fig. 72

r = radius of curvature of the rib,
dc = elementary fibre length at a distance of y from the neutral axis,
$d\phi$ = elementary central angle,

so that

$$dc = (r + y)\, d\phi.$$

Further, let

f = fibre stress at y,
$\Delta\, dc$ = deformation of dc due to f,
$\Delta\, d\phi$ = change of $d\phi$,

so that

$$\Delta dc = y \Delta d\phi.$$

Then we have,

$$f = E\frac{\Delta dc}{dc} = E\frac{y\Delta\, d\phi}{(r + y)\, d\phi}.$$

Since

$$M = \int fydA,$$

substituting the value of f,

$$M = E\frac{\Delta\, d\phi}{d\phi}\int\frac{y^2 dA}{r + y},$$

or

$$f = \frac{My}{(r + y)\int\frac{y^2 dA}{r + y}}.$$

From the last equation it will be seen that

$$f = \frac{My}{\int y^2 dA} = \frac{My}{I}$$

only when y may be neglected.

CHAPTER VI

SUSPENSION BRIDGES

99. SUSPENSION bridges are lacking in rigidity besides being expensive of construction, and for that reason, those of moderate spans are but rarely constructed. It suffices to state that suspension bridges with trussed links are nothing more than inverted arches, and are to be treated in almost exactly the same manner as the latter of corresponding forms, with change in signs of stresses, and taking into consideration the motion of supports. This last condition brings the following term into the expression for work of resistance,

$$\tfrac{1}{2}H\Delta l,$$

in which Δl represents the change in span length due to the motion of supports caused by the change in length of the anchor ties. In the suspension bridge of Fig. 73, we have,

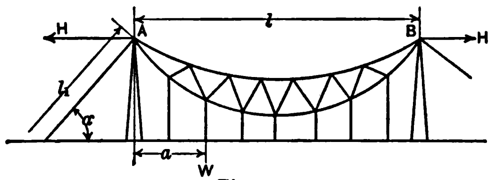

Fig. 73

$$\Delta l = 2\Delta l_1 \sec \alpha.$$

Neglecting the frictional resistance to the motion of saddles on the towers or of the pin (in case the towers are hinged at the base), we get,

$$\Delta l_1 = \frac{Hl_1 \sec \alpha}{EA_a},$$

in which A_a represents the sectional area of the anchor tie. Whence

$$\tfrac{1}{2} H \Delta l = \frac{H^2 l_1 \sec^2 a}{EA_a}.$$

Neglecting the compression of towers, we have, by referring to Art. 45, the following expression for H due to one load W:

$$H = \frac{1}{2} \frac{\int_0^{a'} \frac{xydc}{I} + a \int_{a'}^{\frac{l'}{2}} \frac{ydc}{I} - \int_0^a \frac{\sin \phi \, dx}{A}}{\int_0^{\frac{l'}{2}} \frac{y^2dc}{I} + \int_0^{\frac{l}{2}} \frac{\cos \phi \, dx}{A} + \frac{l_1 \sec^2 a}{A_a}} W. \quad . \quad . \quad . \quad (215)$$

For uniform temperature change t, since the actual horizontal displacements of A and B amount to

$$2\, t\theta l_1 \sec a,$$

we have, by referring to Art. 46, the following equation:

$$H_t = \frac{t\theta E\,(l + 2\, l_1 \sec a)}{\int_0^{l'} \frac{y^2dc}{I} + \int_0^{l} \frac{\cos \phi \, dx}{A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (216)$$

100. For a long span, which is the proper field for a suspension bridge, the latter, as generally constructed, consists of cables and stiffening trusses, forming a composite system of construction.

101. Cable Suspension Bridge with Continuous Stiffening Trusses.— No mathematically correct method of calculating stresses in this kind of suspension bridge has, as yet, been made practicable, — most formulas used in practice being more or less rough approximations. The following method of calculation is another of the latter kind.

The cable — being given a uniform cross-section — forms, under its own weight, a catenary curve; since, however, the weight of the floor system and stiffening trusses, which is generally uniformly distributed along the horizontal line, is far in excess of that of the cable itself, the curve actually assumed by the latter is closely

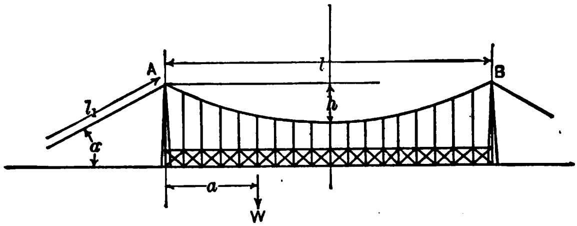

Fig. 74

allied to a parabola, especially as the suspender rods are so adjusted as to produce this form as nearly as possible. The entire weight of the stiffening girder in normal condition and at mean temperature, when not loaded, may thus be assumed to be borne by the cable, the ends of the girder touching the supports but producing no reaction. If now a load W be placed on the girder, the elongation of the cable and suspenders will take place, and in consequence the stiffening girder will bear on the supports. A part of W will then be borne by the stiffening girder and be transmitted directly to the supports, while the remainder goes through suspenders to the cable. For simplicity of calculation, it will be assumed in the following that the tension in suspenders due to any loading will be uniform, as in the case of dead loads. Such an assumption — although never strictly correct — is permissible considering the considerable rigidity generally given to the stiffening trusses.

Let

p = the pull in the suspenders per unit length of the span.
H = the horizontal component of tension in the cable due to p.
x, y = coördinates with origin at A.
c = length measured along the cable.
l = the distance between the towers.
l' = the total length of the cable between the towers.
l_1 = the length of the anchor cable.
A_c = the cross-section of the cables.
I = the moment of inertia of the stiffening trusses.

Referring, then, to Fig. 74, we get the following works of resistance:

(1) Work in the cable. Passing a section through the centre of the span, and taking moment at A or B, we have,

$$H = \frac{pl^2}{8\,h}.$$

Since p is assumed to be acting vertically, H will be constant throughout. The stress in the cable at any point x will then be,

$$H\frac{dc}{dx},$$

and the work of resistance due to the same,

$$2\int_0^{\frac{l'}{2}} \frac{\left(H\dfrac{dc}{dx}\right)^2 dc}{2\,A_cE}.$$

Since

$$dc = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \frac{dy^2}{dx^2}},$$

introducing in this, the equation of parabola

$$y = \frac{4\,h}{l^2}\,x\,(l - x),$$

we get,

$$dc = \left\{1 + \frac{8\,h^2}{l^4}\,(l - 2\,x)^2\right\}dx, \text{ nearly.}$$

Substituting and integrating, we get for the total work of resistance in the cable the following approximate expression:

$$\frac{H^2\,(l^2 + 8\,h^2)}{2\,A_c El},$$

or

$$\frac{\left(\frac{pl^2}{8\,h}\right)^2 (l^2 + 8\,h^2)}{2\,A_c El}.$$

(2) Work in anchor cables. If we neglect the resistance offered to the motion of the saddle, the tension in the anchor cable would be,

$$H \sec \alpha,$$

and the work of resistance in the same,

$$2\int_0^{l_1} \frac{(H \sec \alpha)^2\,dc}{2\,EA_c} = \frac{p^2 l^4 l_1 \sec^2 \alpha}{64\,EA_c h^2}.$$

(3) Work in suspenders. Denoting with L the mean length of all the suspenders, and with A_s their total cross-sectional area, we at once get for the work in the suspenders the following expression:

$$\frac{(pl)^2\,L}{2\,A_s E}.$$

This term is generally so small compared with others, forming the total work of resistance, that it may be entirely neglected without appreciable error.

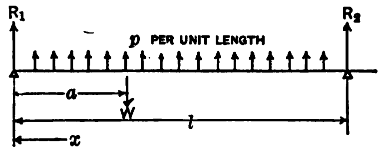

Fig. 75

(4) Work in stiffening trusses. Let R_1 and R_2 represent the end reactions (Fig. 75) of the stiffening truss. Taking moments at each support successively, we get,

$$R_1 = \frac{W(l-a)}{l} - \frac{pl}{2},$$

$$R_2 = \frac{Wa}{l} - \frac{pl}{2},$$

also

$$R_1 + R_2 + pl = W.$$

At any point distant x from the left end, we have for moment,

$$m = R_1 x + \frac{px^2}{2} = \frac{W(l-a)}{l}x - \frac{px}{2}(l-x) \quad \text{for } x < a.$$

$$m = R_1 x + \frac{px^2}{2} - W(x-a) = \frac{Wa(l-x)}{l} - \frac{px(l-x)}{2} \text{ for } x > a.$$

Neglecting the deformation of web-members, we get for work of resistance, the following expression:

$$\int_0^l \frac{m^2 dx}{2IE} = \int_0^a \frac{\left\{\frac{W(l-a)}{l}x - \frac{px(l-x)}{2}\right\}^2 dx}{2IE} + \int_a^l \frac{\left\{\frac{Wa(l-x)}{l} - \frac{px(l-x)}{2}\right\}^2 dx}{2IE}.$$

Assuming I and E to be uniform throughout, we get for the integrals,

$$\frac{40\,W^2a^2(l-a)^2 - 10\,Wpla\,(l^3 - 2\,a^2l + a^3) + p^2l^6}{240\,IEl}.$$

If we neglect the deformations produced in the towers and the anchorages, we have for the total work of resistance in the structure,

$$\omega = \frac{p^2l^3(l^2 + 8\,h^2)}{128\,h^2A_cE} + \frac{p^2l^4l_1 \sec^2 \alpha}{64\,A_cEh^2}$$
$$+ \frac{40\,W^2a^2(l-a)^2 - 10\,Wpla\,(l^3 - 2\,a^2l + a^3) + p^2l^6}{240\,IEl}.$$

Setting the first derivative of ω with respect to p equal to zero, we get,

$$p = \frac{a\,(l^3 - 2\,a^2l + a^3)}{\frac{3\,l^3I}{8\,h^2A_c}(l^2 + 8\,h^2 + 2\,ll_1 \sec^2 \alpha) + \frac{l^5}{5}}\,W \quad . \quad . \quad (217)$$

For a uniform load w per unit length, we have but to put wda instead of W, and integrate between given limits of loading. Thus, for the load extending for a_1 from the left support, we have,

$$p = \frac{a_1^2\,(5\,l^3 - 5\,a_1^2l + 2\,a_1^3)\,w}{\frac{15\,l^3I}{4\,h^2A_c}(l^2 + 8\,h^2 + 2\,ll_1 \sec^2 \alpha) + 2\,l^5} \quad . \quad . \quad (218)$$

102. The following stresses may now be written:
The *stress in the cable* — maximum at the towers,

$$H \sec \phi_0 = \frac{1}{8}\frac{pl^2}{h} \sec \phi_0.$$

ϕ_0 being the inclination of the tangent at towers to the horizontal.

The *stress in the anchor cable,*

$$\frac{1}{8}\frac{pl^2}{h} \sec \alpha, \text{ approximately.}$$

The *stress in the suspender,*

$$\frac{pl}{n},$$

n denoting the number of panels into which equi-distant suspenders divide the span.

In the *stiffening truss* at any point distant x from the left end,

$$\left.\begin{aligned}\text{Shear} &= R_1 + px & \text{for } x < a \\ &= R_1 + px - W & \text{for } x > a\end{aligned}\right\} \text{due to } W.$$

$$\left.\begin{aligned}&= R_1 + (p - w)\,x & \text{for } x < a_1 \\ &= R_1 + px - wa_1 & \text{for } x > a_1\end{aligned}\right\} \text{due to } w.$$

$$\left.\begin{aligned}\text{Moment} &= R_1x + \frac{px^2}{2} & \text{for } x < a \\ &= R_1x + \frac{px^2}{2} - W(x - a) & \text{for } x > a\end{aligned}\right\} \text{due to } W.$$

$$\left.\begin{aligned}&= R_1x + \frac{(p - w)}{2}x^2 & \text{for } x < a_1 \\ &= R_1x + \frac{px^2}{2} - a_1w\left(x - \frac{a_1}{2}\right) & \text{for } x > a_1\end{aligned}\right\} \text{due to } w.$$

103. Temperature Stresses. — It hardly requires explanation that a rising temperature tends to strain the stiffening trusses and at the same time relieve the cables, while falling temperature strains both cables and trusses.

Since,

$$l' = \int_0^{l'} dc = l\left(1 + \frac{8h^2}{3l^2}\right), \text{ nearly (see page 143),}$$

a uniform change of t degrees—positive for rise—changes the length of the main-span cable by

$$t\theta l' = t\theta l\left(1 + \frac{8\,h^2}{3\,l^2}\right), \text{ nearly.}$$

Representing by h_1 the sag of the deformed cable, we have,

$$l\,(1 + t\theta)\left(1 + \frac{8\,h^2}{3\,l^2}\right) = l\left(1 + \frac{8\,h_1^2}{3\,l^2}\right),$$

from which

$$h_1 = h\left\{1 + \frac{t\theta}{2}\left(1 + \frac{3\,l^2}{8\,h^2}\right)\right\}, \text{ nearly.}$$

The deformed cable would then deflect upward or downward by about

$$\frac{ht\theta}{2}\left(1 + \frac{3}{8}\,\frac{l^2}{h^2}\right) = \frac{3\,t\theta l^2}{16\,h}, \text{ approximately.}$$

The change in the length of anchor cables due to t being $2\,t\theta l_1$, will change the main-span length by

$$-\,2\,t\theta l_1 \sec \alpha.$$

Representing by h_2, the sag of the cable with changed span length, we have,

$$(l - 2\,\theta t l_1 \sec \alpha)\left\{1 + \frac{8\,h_2^2}{3\,(l - 2\,\theta t l_1 \sec \alpha)^2}\right\} = l\left(1 + \frac{8\,h^2}{3\,l^2}\right),$$

so that the deflection of the cable due to this cause alone would be

$$\frac{3}{8}\,\frac{l}{h}\,\theta t l_1 \sec \alpha, \text{ approximately.}$$

The total deflection of the cable, due to a temperature change of t, were the latter free to deflect, would then be,

$$\delta = \frac{3}{16}\,\frac{l}{h}\,t\theta\,(l + 2\,l_1 \sec \alpha)\;.\;\;.\;\;.\;\;.\;\;. \qquad (219)$$

This, however, could not take place owing to the rigidity of the truss; but as the difference is generally slight, we shall assume δ to be the total deflection.

Neglecting the changes in lengths of suspenders, and further assuming the stress produced in the latter by the temperature change to be uniform, and calling it p_t per unit length of the span, we get,

$$p_t = \frac{384\,EI}{5\,l^4}\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (220)$$

This uniform pull on the cable produces tension in the latter, whose horizontal component is

$$H_t = \frac{48\,EI}{5\,hl^2}\delta,$$

and in the stiffening truss a moment of

$$\frac{48\,EI}{5\,l^2}\delta$$

at the centre of the span.

Each suspender sustains a pull due to this cause of

$$\frac{p_t l}{n}.$$

104. There is still another factor, not considered in the preceding discussion, viz.: the change in the amount of H due to variation of the deflection of the cable from whatever cause. Thus, if at the normal temperature

$$H = \frac{pl^2}{8\,h},$$

a change of t degrees would give

$$H = \frac{pl^2}{8\,h\left\{1 + \frac{t\theta}{2}\left(1 + \frac{3}{8}\frac{l^2}{h^2}\right)\right\}} \quad . \quad . \quad . \quad . \quad (221)$$

TRUSSES WITH REDUNDANT MEMBERS

105. The different forms of trusses with redundant members may be indefinitely multiplied. They are generally decomposable into as many simple trusses as there are systems of such members to form them. The following few simple cases will sufficiently explain a mode of procedure for calculating the stresses in this kind of trusses.

106. Fig. 76 shows an ordinary lattice truss. The

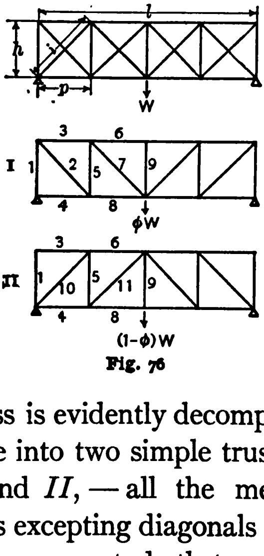

Fig. 76

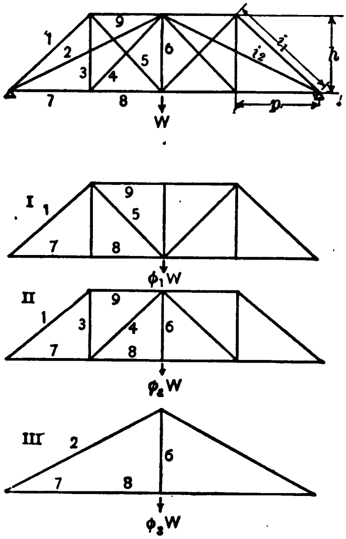

Fig. 77

truss is evidently decomposable into two simple trusses *I* and *II*, — all the members excepting diagonals being common to both trusses.

A load W hung at the middle panel point must — according to the principle of least work — be divided between trusses *I* and *II* in such a way that the internal work performed in the truss will be a minimum.

Representing by ϕ the ratio of the load borne by truss I to the whole, we have the following stresses:

Members.	1	2	3	4	5	6
Trusses. I	$-\frac{1}{2}\phi W$	$+\frac{i}{2h}\phi W$	$-\frac{p}{2h}\phi W$	o	$-\frac{1}{2}\phi W$	$-\frac{p}{h}\phi W$
II	o	. . .	o	$+\frac{p}{2h}(1-\phi)W$	$-\frac{1}{2}(1-\phi)W$	$-\frac{p}{2h}(1-\phi)W$

Members.	7	8	9	10	11
Trusses. I	$+\frac{i}{2h}\phi W$	$+\frac{p}{2h}\phi W$	o	. . . . .	. . . .
II	. . .	$+\frac{p}{h}(1-\phi)W$	$+(1-\phi)W$	$-\frac{i}{2h}(1-\phi)W$	$-\frac{i}{2h}(1-\phi)W$

Denoting by A with corresponding suffixes the cross-sectional areas of members, we get for the internal work in the trusses the following:

$$\frac{\Sigma S^2L}{2AE}=\frac{W^2}{4Eh^2}\left\{\phi^2\left(\frac{h^3}{A_1}+\frac{i^3}{A_2}+\frac{i^3}{A_7}+\frac{p^3}{A_3}\right)+(1-\phi)^2\left(\frac{2h^3}{A_9}+\frac{i^3}{A_{10}}+\frac{i^3}{A_{11}}+\frac{p^3}{A_4}\right)\right.$$
$$\left.+\frac{h^3}{A_5}+(1+\phi)^2\frac{p^3}{A_6}+(2-\phi)^2\frac{p^3}{A_8}\right\}=\omega.$$

Since the value of ϕ must be such as to make the internal work a minimum, differentiating ω with respect to ϕ, and setting the differential coefficient equal to zero, we at once obtain,

$$\phi=\frac{h^3\left(\frac{2}{A_9}\right)+i^3\left(\frac{1}{A_{10}}+\frac{1}{A_{11}}\right)+p^3\left(\frac{1}{A_4}-\frac{1}{A_6}+\frac{2}{A_8}\right)}{h^3\left(\frac{1}{A_1}+\frac{2}{A_9}\right)+i^3\left(\frac{1}{A_2}+\frac{1}{A_7}+\frac{1}{A_{10}}+\frac{1}{A_{11}}\right)+p^3\left(\frac{1}{A_3}+\frac{1}{A_4}+\frac{1}{A_6}+\frac{1}{A_8}\right)}$$

107. Fig. 77 shows another case of a truss with redundant members, formed evidently by a combination of Pratt, Howe and King-post trusses. In order to find the stresses in different members of the truss, decompose the latter, as before, into elementary trusses *I*, *II* and *III*, and denote by ϕ_1, ϕ_2 and ϕ_3 the ratios of distribution of W in *I*, *II* and *III* respectively, so that

$$\phi_1 + \phi_2 + \phi_3 = 1.$$

The following stresses may now be written:

Members.	1	2	3	4	5
Trusses. I	$-\frac{i_1}{2h}\phi_1 W$	. . .	. . .	. . . .	$+\frac{i_1}{2h}\phi_1 W$
II	$-\frac{i_1}{2h}\phi_2 W$	. . .	$+\frac{1}{2}\phi_2 W$	$-\frac{i_1}{2h}\phi_2 W$	. . . .
III	. . . .	$-\frac{i_2}{2h}\phi_3 W$	. . .	. . .	. . . .

Members.	6	7	8	9	
Trusses. I	. . . .	$+\frac{p}{2h}\phi_1 W$	$+\frac{p}{2h}\phi_1 W$	$-\frac{p}{h}\phi_1 W$	. . . .
II	$+\phi_2 W$	$+\frac{p}{2h}\phi_2 W$	$+\frac{p}{h}\phi_2 W$	$-\frac{p}{2h}\phi_2 W$	. . . .
III	$+\phi_3 W$	$+\frac{p}{h}\phi_3 W$	$+\frac{p}{h}\phi_3 W$	. . . .	. . . .

Denoting by A with suffixes the cross-sections of the members, we get for ω,

$$\frac{\Sigma S^2L}{2AE}=\frac{W^2}{2EH^2}\left\{\frac{i_1^3(\phi_1+\phi_2)^2}{A_1}+\frac{i_2^3(1-\phi_1-\phi_2)^2}{A_2}+\frac{h^3\phi_2^2}{A_3}+\frac{i_1^3\phi_2^2}{A_4}+\frac{i_1^3\phi_1^2}{A_5}\right.$$
$$\left.+\frac{h^3(1-\phi_1)^2}{A_6}+\frac{p^3(2-\phi_1-\phi_2)^2}{A_7}+\frac{p^3(2-\phi_1)^2}{A_8}+\frac{p^3(2\,\phi_1+\phi_2)^2}{A_9}\right\}=\omega.$$

Making

$$\frac{d\omega}{d\phi_1}=0,$$
$$\frac{d\omega}{d\phi_2}=0,$$

we obtain at once the following equations:

$$\phi_1\left\{i_1^3\left(\frac{1}{A_1}+\frac{1}{A_5}\right)+\frac{i_2^3}{A_2}+\frac{2\,h^3}{A_6}+p^3\left(\frac{1}{A_7}+\frac{1}{A_8}+\frac{4}{A_9}\right)\right\}+\phi_2\left\{\frac{i_1^3}{A_1}+\frac{i_2^3}{A_2}\right.$$
$$\left.+p^3\left(\frac{1}{A_7}+\frac{2}{A_9}\right)\right\}-\left\{\frac{i_2^3}{A_2}+\frac{2\,h^3}{A_6}+2\,p^3\left(\frac{1}{A_7}+\frac{1}{A_8}\right)\right\}=0,$$

$$\phi_2\left\{\frac{i_1^3}{A_1}+\frac{i_2^3}{A_2}+p^3\left(\frac{1}{A_7}+\frac{2}{A_9}\right)\right\}+\phi_2\left\{i_1^3\left(\frac{1}{A_1}+\frac{1}{A_4}\right)+\frac{i_2^3}{A_2}+p^3\left(\frac{1}{A_7}+\frac{2}{A_9}\right)\right\}$$
$$-\left(\frac{i_2^3}{A_2}+\frac{2\,p^3}{A_7}\right)=0.$$

Representing the terms within the brackets by a, β, γ, δ and ϵ in order, we get

$$\phi_1 a+\phi_2\beta=\gamma,$$
$$\phi_1\beta+\phi_2\delta=\epsilon,$$

from which

$$\phi_1=\frac{\beta\epsilon-\gamma\delta}{\beta^2-a\delta},$$
$$\phi_2=\frac{\beta\gamma-a\epsilon}{\beta^2-a\delta},$$
$$\phi_3=\frac{\beta(\beta-\gamma)+\epsilon(a-\beta)+\delta(\gamma-a)}{\beta^2-a\delta}.$$

As this chapter is the least important of all, considering the comparative rarity of the kinds of structures treated, it is deemed sufficient merely to indicate a mode of procedure in the calculation of stresses by means of the method of work.

CHAPTER VII

SECONDARY STRESSES DUE TO THE RIGIDITY OF JOINTS

108. Strictly speaking, the stresses in all kinds of trusses are statically indeterminate; for it is only by assuming joints to be free of all constraints under all changes of form due to elastic deformation of members that we are generally enabled to calculate the stresses in trusses when external forces are known, while in reality all joints are more or less rigid and resistant to angular changes between truss-members. Even pin-connected trusses are to be considered to have rigid joints wherever the frictional resistance on the surface of the pin exceeds the torsional moment caused in the latter by the members which it connects.

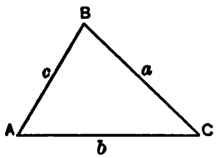

Fig. 78

In the following, we shall in the first place discuss the secondary stresses in the truss itself, and then those in the posts and lateral members due to rigid floor-beam and lateral connections.

SECONDARY STRESSES IN TRUSSES

109. Suppose a truss element ABC (Fig. 78) undergo deformations in its sides a, b and c due to stresses produced in them. If the joints do not offer any resistance to motion of members, then corresponding angular changes would take place forming a new triangle with changed side-lengths. Representing by Δ the rate of linear and angular changes, + for extension or enlargement and − for the contrary, we have the following equations from the deformed triangle:

$$(b + \Delta b) \sin (A + \Delta A) = (a + \Delta a) \sin (B + \Delta B),$$

from which, by neglecting small quantities, we get

$$\Delta A = \left(\frac{\Delta a}{a} - \frac{\Delta b}{b}\right) \tan A + \Delta B \frac{\tan A}{\tan B}.$$

Similarly,

$$\Delta C = \left(\frac{\Delta c}{c} - \frac{\Delta b}{b}\right) \tan C + \Delta B \frac{\tan C}{\tan B};$$

and since

$$\Delta A + \Delta B + \Delta C = 0,$$

we get,

$$\Delta B = \frac{\Delta b}{b} (\cot A + \cot C) - \frac{\Delta a}{a} \cot C - \frac{\Delta c}{c} \cot A,$$

$$\Delta C = \frac{\Delta c}{c} (\cot A + \cot B) - \frac{\Delta a}{a} \cot B - \frac{\Delta b}{b} \cot A,$$

$$\Delta A = \frac{\Delta a}{a} (\cot B + \cot C) - \frac{\Delta b}{b} \cot C - \frac{\Delta c}{c} \cot B.$$

These equations would then give the amounts of angular changes due to changes in lengths of the members were the latter free to move at the joints. If, however, the joints were so rigid that angular changes could not take place, then each member would have to distort itself in such a way as to occupy its new position without producing angular changes at the joints. As a consequence, whatever positions the members may in this way assume, the tangents to their neutral axes at the joints should maintain the original angles between them. The constraint thus made active at each joint gives rise to bending in each member. Let a member AC be cut off close to the joints (Fig. 79), and represent by M_a and M_c the moments acting at the ends, — positive for bending the piece anticlockwise and vice versâ. Further, let AC' represent the original position of AC, the former being tangent to the latter at A; and denote by α the displacement of C, — positive when it is clockwise and vice versâ.

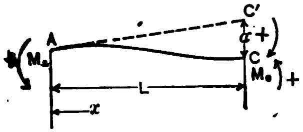

Fig. 79

Since the moment (positive when causing compression in upper fibres) at any point distant x from A is

$$-M_a + \frac{M_a + M_c}{L}x,$$

considering the moments only, we get for the internal work in the member,

$$\omega = \frac{1}{2EI}\int_0^L \left(-M_a + \frac{M_a + M_c}{L}x\right)^2 dx$$

$$= \frac{L}{6EI}(M_a^2 - M_aM_c + M_c^2),$$

in which I represents the moment of inertia of the section of the piece, and E the modulus of elasticity, both assumed to be uniform.

Since the force acting through α, reckoned in the same sense as the latter, is

$$-\frac{M_a + M_c}{L},$$

it follows from the first theorem of Castigliano (Art. 6) that the first derivative of ω with respect to $\frac{M_a + M_c}{L}$ must be equal to a, so that with M_a as variable we have,

$$\frac{L}{6\,EI}(2\,M_a - M_c) = \frac{a}{L} \quad . \quad . \quad . \quad . \quad (222)$$

Considering now the joint A (Fig. 80), and applying this equation to members AC and AB — denoting their lengths, end-displacements and moments by L, a, and M with corresponding suffixes as shown in the figure — we get,

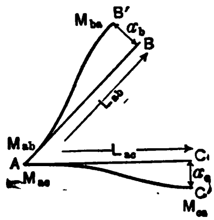

Fig. 80

$$\frac{L_{ac}}{6\,EI_{ac}}(2\,M_{ac} - M_{ca}) = \frac{a_c}{L_{ac}} \quad . \quad . \quad . \quad . \quad (223)$$

$$\frac{L_{ab}}{6\,EI_{ab}}(2\,M_{ab} - M_{ba}) = \frac{a_b}{L_{ac}} \quad . \quad . \quad . \quad . \quad (224)$$

Referring to Fig. 80, it will be seen that the angular change ΔA, which would have taken place had the angle A been free to change, to be equal to $BAC - B'AC'$, so that by paying attention to signs we may put

$$\Delta\,(BAC) = \frac{a_c}{L_{ac}} - \frac{a_b}{L_{ab}}.$$

Consequently we get,

$$6\,E\,\Delta\,BAC = \frac{L_{ac}}{I_{ac}}(2\,M_{ac} - M_{ca}) - \frac{L_{ab}}{I_{ab}}(2\,M_{ab} - M_{ba})\,. \quad . \quad . \quad (225)$$

Similarly for other angles, we obtain,

$$6\,E\,\Delta\,ABC = \frac{L_{ab}}{I_{ab}}(2\,M_{ab} - M_{ba}) - \frac{L_{bc}}{I_{bc}}(2\,M_{bc} - M_{cb}) \quad . \quad . \quad . \qquad (226)$$

$$6\,E\,\Delta\,BCA = \frac{L_{cb}}{I_{cb}}(2\,M_{cb} - M_{bc}) - \frac{L_{ca}}{I_{ca}}(2\,M_{ca} - M_{ac}) \quad . \quad . \quad . \qquad (227)$$

Since at each joint

$$\Sigma\,M = 0,$$

we have in the triangular frame *ABC*,

$$M_{ab} + M_{ac} = 0,$$
$$M_{ba} + M_{bc} = 0,$$
$$M_{cb} + M_{ca} = 0.$$

In this way, as many equations as there are unknown moments in a truss may be obtained by extending equations over all the triangles of the truss and setting the $\Sigma\,M$ at each joint equal to zero. The following example will show the application of this method to the calculation of secondary stresses in a simple truss.

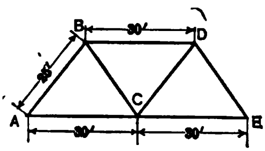

Fig. 81

Example. — In the truss of Fig. 81 given:

$I_{ab} = 360$ in.^{4}	$L_{ab} = L_{bc} = 25$ ft.
$I_{bc} = 60$ "	$L_{bd} = L_{ac} = 30$ "
$I_{ac} = 40$ "	
$I_{bd} = 360$ "	

Supposing the members to sustain following intensities of direct stresses:

$\overline{AB} = \overline{DE} = -\ 7000$ lbs. per sq. in.

$\overline{AC} = \overline{CE} = +\ 7330$ lbs. per sq. in.

$\overline{BD} = -\ 7330$ lbs. per sq. in.

$\overline{BC} = \overline{CD} = +\ 8000$ lbs. per sq. in.

to calculate the secondary stresses induced in each member, assuming the joints to be perfectly rigid, and E taken at 30,000,000 lbs. per sq. in.

In consequence of the principal stresses the following deformations are produced:

In AB and DE . $\Delta L = -\frac{7000}{E} \times 25 \times 12 = -.070$ in.

AC and CE . $\Delta L = \frac{7330}{E} \times 30 \times 12 = +.088$ in.

BD . . . . . $\Delta L = \frac{-7330}{E} \times 30 \times 12 = -.088$ in.

BC and CD . $\Delta L = \frac{8000}{E} \times 25 \times 12 = +.080$ in.

Since

$$\cot BAC = .7500,$$
$$\cot ABC = .2915,$$
$$\cot BCA = .7500,$$

the angular changes, were they possible, would then be,

$$\Delta BAC = \frac{.080}{300}(.2915+.7500) - \frac{.088}{360} \times .75 + \frac{.07}{300} \times .2915 = .0001624,$$

$$\Delta ABC = \frac{.088}{360}(.7500+.7500) - \frac{.08}{300} \times .75 + \frac{.07}{300} \times .7500 = .0003416,$$

$$\Delta BCA = -\frac{.070}{300}(.7500+.2915) - \frac{.08}{300} \times .2915 - \frac{.088}{360} \times .7500$$
$$= -.0005040,$$

$$\Delta DBC = \frac{.080}{300}(.7500+.2915) - \frac{.08}{300} \times .2915 + \frac{.088}{360} \times .7500 = .0003835,$$

$$\Delta BCD = -\frac{.088}{360}(.7500+.7500) - \frac{.80}{300} \times .7500 - \frac{.08}{300} \times .7500$$
$$= -.0007670.$$

Applying Eqs. (225–227) to all the angles successively, we obtain,

$$1)\quad 6E\,.0001624 = \frac{360}{40}(2M_{ac} - M_{ca}) - \frac{300}{360}(2M_{ab} - M_{ba}).$$

$$2)\quad 6E\,.0003416 = \frac{300}{360}(2M_{ba} - M_{ab}) - \frac{300}{60}(2M_{bc} - M_{cb}).$$

$$3)\; -6E\,.0005040 = \frac{300}{60}(2M_{cb} - M_{bc}) - \frac{360}{40}(2M_{ca} - M_{ac}).$$

$$4)\quad 6E\,.0003835 = \frac{300}{60}(2M_{bc} - M_{cb}) - \frac{360}{360}(2M_{bd} - M_{db}).$$

$$-\,6E\,.0007670 = \frac{300}{60}(2M_{cd} - M_{dc}) - \frac{300}{60}(2M_{cb} - M_{bc}).$$

Since from symmetry,

$$M_{cd} = -M_{cb}, \qquad M_{dc} = -M_{bc}, \qquad M_{db} = -M_{bd},$$

the last equation becomes

$$5)\quad 6E\,.0007670 = \frac{600}{60}(2M_{cb} - M_{bc}).$$

Again, since, at each joint, $\Sigma M = 0$,

$$6)\quad M_{ab} + M_{ac} = 0.$$

$$7)\quad M_{ba} + M_{bc} + M_{bd} = 0.$$

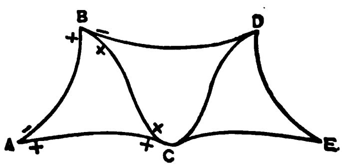

Fig. 82

From these equations we obtain the following values of Ms:

$M_{ac} = +\ 2,602$ in.-lbs.
$M_{ab} = -\ 2,602$ "
$M_{ba} = +\ 25,926$ "
$M_{bc} = +\ 2,454$ "
$M_{bd} = -\ 28,380$ "
$M_{cb} = +\ 8,130$ "
$M_{ca} = +\ 10,176$ "

Paying attention to the signs of the moments, it will be seen that the bending of the members takes place as shown exaggerated in Fig. 82.

The widths of members in the plane of the truss being as follows,

$$AB = 12 \text{ in.},$$
$$BD = 12 \text{ "}$$
$$BC = 4 \text{ "}$$
$$AC = 4 \text{ "}$$

the secondary stress in each member may now be written,

In AB . . . . . $\frac{25926}{360} \times 6 = 432$ lbs. per sq. in.

BD . . . . . $\frac{28380}{360} \times 6 = 473$ " " " "

BC . . . . . $\frac{8130}{60} \times 2 = 271$ " " " "

AC . . . . . $\frac{10176}{40} \times 2 = 509$ " " " "

These stresses are to be added to the direct stresses already given, to obtain the maximum intensities of stresses in the members. Strictly speaking, the actual maximum in each member is less than the maximum thus found, by that portion of the direct stress taken up in producing the moments, but the difference is generally so slight as to be practically negligible.

SECONDARY STRESS IN THE POSTS DUE TO RIGID FLOOR-BEAM AND LATERAL CONNECTIONS

110. By Loading. — When the posts are connected rigidly at top by the lateral strut, and at bottom by the floor-beam, — as is most usually the case, — the bending of the floor-beam caused by loading on the same induces — by reason of constraints — the moments and thrust, producing deformations as shown exaggerated in Fig. 83.

Representing by M_0, M_1, M_2, and M_3 (Fig. 84) the moments produced at the four corners of the frame, — positive when producing compression on the outer fibre,

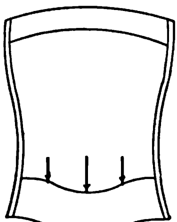

Fig. 83

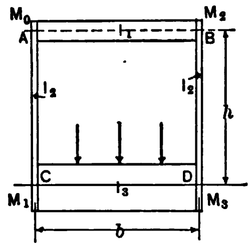

Fig. 84

and vice versâ, — it is evident that, for all loads symmetrically disposed about the centre of the floor-beam,

$$M_0 = M_2,$$
$$M_1 = M_3.$$

Let m_1, m_2, and m_3 represent moments at any point in the upper strut AB, post AC, and floor-beam CD respectively. Then, neglecting the influence of direct stresses and considering moments alone, we get for the work of resistance in the frame,

$$\omega = \int_0^b \frac{m_1^2dx}{2EI_1} + 2\int_0^h \frac{m_2^2dx}{2EI_2} + \int_0^b \frac{m_3^2dx}{2EI_3},$$

I_1, I_2, and I_3 representing the moments of inertia of the sections of strut, post, and floor-beam respectively. Since

$$m_1 = M_0,$$
$$m_2 = M_0 + \frac{M_1 - M_0}{h} x, \text{ origin of } x \text{ taken at } A,$$
$$m_3 = M_1 + M,$$

in which M represents the moment due to loading, assuming the floor-beam to be simply supported at both ends.

Substituting these values of m in the above expression for work, and integrating by assuming the cross-sections of members to be so many constants, we get,

$$\omega = \frac{M_0^2 b}{2 E I_1} + \frac{3 M_0 M_1 h + (M_1 - M_0)^2 h}{3 E I_2}$$
$$+ \frac{1}{2 E I_3}\left(M_1^2 b + 2 M_1 \int_0^b M dx + \int_0^b M^2 dx\right).$$

Since the values of M_0 and M_1 must be such as to make the total work of resistance in the frame a minimum, setting the first derivatives of ω taken successively with respect to M_0 and M_1 equal to zero, we get,

$$\frac{M_0 b}{I_1} + \frac{M_1 h + 2 M_0 h}{3 I_2} = 0,$$
$$\frac{2 M_1 h + M_0 h}{3 I_2} + \frac{M_1 b}{I_3} + \int_0^b \frac{M dx}{I_3} = 0;$$

from which

$$M_0 = -\frac{\frac{h}{3 I_2}}{\frac{b}{I_1} + \frac{2h}{3 I_2}} M_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (228)$$

Eliminating M_1,

$$M_0 = \frac{\frac{h}{3 I_2} \int_0^b \frac{M dx}{I_3}}{\left(\frac{b}{I_1} + \frac{2h}{3 I_2}\right)\left(\frac{2h}{3 I_2} + \frac{b}{I_3}\right) - \left(\frac{h}{3 I_2}\right)^2} \quad . \quad . \quad . \quad . \quad (229)$$

111. The integral $\int_0^b Mdx$ depends on the mode of symmetrical loading. For a *single-track railway bridge* loaded as shown in Fig. 85, we have,

$$\int_0^b Mdx = -2\int_0^{\frac{b-a}{2}} Wxdx - \int_{\frac{b-a}{2}}^{\frac{b+a}{2}} W\left(\frac{b-a}{2}\right)dx = -\frac{W}{4}(b^2-a^2),$$

so that we get,

$$M_0 = -\frac{\frac{h}{12 I_2 I_3}(b^2-a^2)}{\left(\frac{b}{I_1}+\frac{2h}{3I_2}\right)\left(\frac{2h}{3I_2}+\frac{b}{I_3}\right)-\left(\frac{h}{3I_2}\right)^2}W. \quad . \quad (230)$$

$$M_1 = \frac{\frac{b^2-a^2}{4I_3}\left(\frac{b}{I_1}+\frac{2h}{3I_2}\right)}{\left(\frac{b}{I_1}+\frac{2h}{3I_2}\right)\left(\frac{2h}{3I_2}+\frac{b}{I_3}\right)-\left(\frac{h}{3I_2}\right)^2}W \quad . \quad . \quad (231)$$

112. In a *highway bridge*, since the live and dead loads may be assumed to be uniformly distributed along the floor-beam length, designating by w the load per unit length, we get,

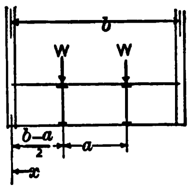

Fig. 85

$$\int_0^b Mdx = -\frac{wb^3}{12};$$

so that from Eqs. (228) and (229) we obtain for this case,

$$M_0 = -\frac{\frac{hb^3}{36 I_2 I_3}w}{\left(\frac{b}{I_1}+\frac{2h}{3I_2}\right)\left(\frac{2h}{3I_2}+\frac{b}{I_3}\right)-\left(\frac{h}{3I_2}\right)^2} \quad . \quad . \quad (232)$$

$$M_1 = \frac{\frac{b^3}{12 I_3}\left(\frac{b}{I_1} + \frac{2h}{3I_2}\right) w}{\left(\frac{b}{I_1} + \frac{2h}{3I_2}\right)\left(\frac{2h}{3I_2} + \frac{b}{I_3}\right) - \left(\frac{h}{3I_2}\right)^2} \quad . \quad . \quad . \quad (233)$$

113. Knowing M_0 and M_1, the moment at any point in each member can at once be found. In the post the maximum fibre stress calculated from such moment is to be combined with the intensity of direct stress which the post receives as the member of the truss, to obtain the greatest intensity of compression in the same.

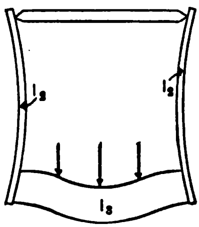

Fig. 86

The *upper strut* is subjected to a compression of

$$\frac{M_1 - M_0}{h},$$

and a bending of M_0 throughout its length. The *floor-beam* sustains, beside a bending of

$$M_1 + M,$$

a direct stress of the same amount but of opposite kind as that in the upper strut. The cross-section of the floor-beam is, however, generally so great that the direct stress produced in the same need hardly ever to be taken into account.

If the upper strut were either hinged at both ends (Fig. 86) or its section so small that I_1 may be entirely neglected, then M_0 would disappear and

$$M_1 = -\frac{\int_0^b \frac{M dx}{I_3}}{\left(\frac{2h}{3I_2} + \frac{b}{I_3}\right)} \quad . \quad . \quad . \quad . \quad . \quad (234)$$

The upper strut would be free of moment, and the direct stress simply equals

$$\frac{M_1}{h}.$$

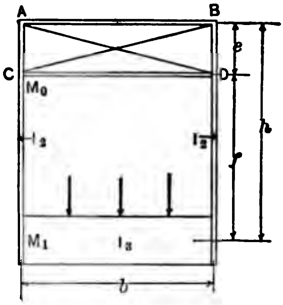

Fig. 87

114. In the bracing of Fig. 87 with designations as given, by neglecting the effect of all direct stresses, we get for the work of resistance,

$$\omega = \frac{1}{EI_2}\int_0^f \left(M_1 + \frac{M_0 - M_1}{f}x\right)^2 dx + \frac{1}{EI_2}\int_0^e \left(\frac{M_0}{e}x\right)^2 dx$$

$$+ \frac{1}{2EI_3}\int_0^b (M_1 + M)^2 dx.$$

Differentiating this with respect to M_0 and M_1 successively, and setting the differential coefficients equal to zero, we get,

$$fM_1 + 2(f + e)M_0 = 0,$$

or

$$M_0 = -\frac{f}{2h}M_1,$$

and

$$\frac{f}{I_2}\left(\frac{M_0}{3} + \frac{2M_1}{3}\right) + \frac{1}{I_3}\left(bM_1 + \int_0^b Mdx\right) = 0.$$

Combining the two equations, we obtain,

$$M_1 = -\frac{\frac{1}{I_3}\int_0 Mdx}{\frac{f(3h+e)}{6hI_2} + \frac{b}{I_3}} \quad . \quad . \quad . \quad . \quad . \quad (235)$$

$$M_0 = \frac{\dfrac{f}{2\,hI_3}\displaystyle\int_0^b Mdx}{\dfrac{f(3\,h+e)}{6\,hI_2} + \dfrac{b}{I_3}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (236)$$

The integral $\int_0^b Mdx$ is the same as in the preceding case.

The stresses in the struts are,

$$\overline{AB} = -\frac{M_0}{e},$$

$$\overline{CD} = \frac{M_0}{e} + \frac{M_0 - M_1}{f}.$$

EXAMPLE. — (1) In a single-track railway bridge with the cross-section of Fig. 88, given:

$$I_1 = 1600 \text{ in.}^4$$
$$I_2 = 800 \text{ in.}^4$$
$$I_3 = 4800 \text{ in.}^4$$
$$W = 60{,}000 \text{ lbs.}$$

Ext. width of post = 15 in.

required to find the maximum fibre stress caused in the posts.

From Eqs. (230) and (231),

$$M_0 = -\frac{\dfrac{20\,(16^2 - 6^2)}{12 \times 800 \times 4800}}{\left(\dfrac{16}{1600} + \dfrac{40}{2400}\right)\left(\dfrac{40}{2400} + \dfrac{16}{4800}\right) - \left(\dfrac{20}{2400}\right)^2} 60{,}000$$

$$= -12{,}350 \text{ ft.-lbs.}$$

$$M_1 = \frac{\dfrac{256 - 36}{4 \times 4800}\left(\dfrac{16}{1600} + \dfrac{40}{2400}\right)}{\dfrac{668}{1{,}440{,}000}} 60{,}000 = 39{,}520 \text{ ft.-lbs.}$$

Consequently the greatest fibre stress in the post is in this case found near to its lower end, and is in amount,

$$\frac{39{,}520 \times 12 \times 7.5}{800} = 3700 \text{ lbs. per sq. in.}$$

(2) In a highway bridge with cross-section as shown in Fig. 89, to calculate the stresses in the post and struts due to

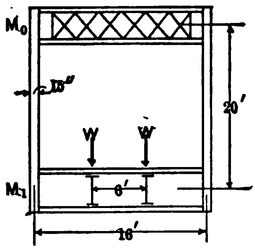

Fig. 88

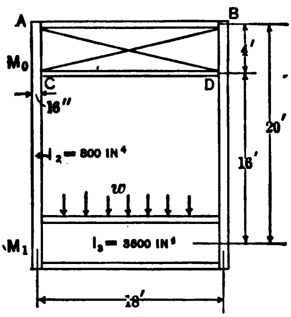

Fig. 89

constraint caused by a uniform load of $w = 3000$ lbs. per ft. run, the dimensions being as given in the figure.

From Eqs. (235) and (236),

$$M_1 = \frac{\dfrac{3000 \times 18^3}{12 \times 3600}}{\dfrac{16\,(60+4)}{6 \times 20 \times 800} + \dfrac{18}{3600}} = +\,25{,}850 \text{ ft.-lbs.}$$

$$M_0 = -\frac{\dfrac{16 \times 3000 \times 18^3}{2 \times 12 \times 20 \times 3600}}{\dfrac{16\,(60+4)}{6 \times 20 \times 800} + \dfrac{18}{3600}} = -\,10{,}340 \text{ ft.-lbs.}$$

The greatest fibre stress in the post will then be,

$$\frac{25{,}850 \times 12 \times 8}{800} = 3102 \text{ lbs. per sq. in.}$$

The stresses in the upper struts are,

$$\overline{AB} = \frac{10,340}{4} = + 2585 \text{ lbs.}$$

$$\overline{CD} = -\frac{10,340}{4} - \frac{10,340 + 25,850}{16} = - 4847 \text{ lbs.}$$

(3) Suppose, in the case of Fig. 88 and Example (1), the upper strut to be hinged at both ends; then from Eq. (234) we get,

$$M_1 = \frac{\frac{W}{4I_3}(b^2 - a^2)}{\frac{2h}{3I_2} + \frac{b}{I_3}} = \frac{\frac{60,000\,(16^2 - 6^2)}{4 \times 4800}}{\frac{2 \times 20}{3 \times 800} + \frac{16}{4800}} = 34,375 \text{ ft.-lbs.},$$

showing that the posts receive somewhat greater stresses from constraint, when the upper strut is firmly fixed to them.

115. By Wind Pressure. — Wind pressure also produces moment and direct stresses in posts forming a rigid frame, whenever the panel wind pressure is to be transferred from one chord to another, — the distortion of the frame being as shown in Fig. 90, in which the wind pressures, acting at the upper panel points are being brought down to the lower. This is simply a somewhat more general case of wind pressure discussed in Art. 20. From what has been there said, it will be easy to see that here again the points of contraflexure in the posts may be assumed to be at the same height.

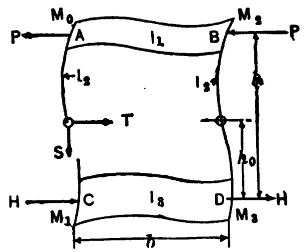

Fig. 90

Assuming the wind pressures $2P$ to be resisted equally at C and D, we have

$$H = P.$$

Passing now a section through the point of contraflexure in the left-side post,.and representing with T and S the tangential and direct stresses acting at the section, we have as before,

$$T = H.$$

Taking moments successively at A, C, O, B, and D, we get,

$$M_0 = -T(h - h_0) = -H(h - h_0),$$
$$M_1 = Hh_0,$$
$$-Sb - 2P(h - h_0) = 0,$$
$$M_2 = -T(h - h_0) - Sb = H(h - h_0) = -M_0,$$
$$M_3 = Th_0 - Sb - 2Ph = -Hh_0 = -M_1.$$

Represent by

m_1 the moment at any point in the upper strut.
m_2 the moment at any point in the post.
m_3 the moment at any point in the floor-beam.

Neglecting the influence of all direct stresses as being inconsiderable when compared with moments, the total work of resistance due to the latter will be,

$$\omega = \int_0^b \frac{m_1^2 dx}{2EI_1} + 2\int_0^h \frac{m_2^2 dx}{2EI_2} + \int_0^b \frac{m_3^2 dx}{2EI_3}.$$

Referring to Fig. 90 and to the preceding equations, we have,

$$m_1 = M_0 + \frac{M_2 - M_0}{b}x = H(h - h_0)\left(\frac{2x}{b} - 1\right), \text{ origin of } x \text{ taken at } A.$$

$$m_2 = M_0 + \frac{M_1 - M_0}{h}x = H(x + h_0 - h), \text{ origin of } x \text{ taken at } A.$$

$$m_3 = M_1 + \frac{M_3 - M_1}{b}x = Hh_0\left(1 - \frac{2x}{b}\right),\text{ origin of } x \text{ taken at } C.$$

Substituting these values of m in the expression for work, and integrating, we get,

$$\omega = \frac{H^2(h-h_0)^2 b}{6EI_1} + \frac{H^2 h(h^2 - 3hh_0 + 3h_0^2)}{3EI_2} + \frac{H^2 h_0^2 b}{6EI_3}.$$

Since the value of h_0 must be such as to make the internal work a minimum, the first derivative of ω with respect to h_0 set equal to zero will at once give,

$$h_0 = \frac{\frac{b}{I_1} + \frac{3h}{I_2}}{\frac{b}{I_1} + \frac{6h}{I_2} + \frac{b}{I_3}}h \quad . \; . \; . \; . \; . \; (237)$$

This value of h_0 substituted in the foregoing equations of moments, will give moments at all points of each member. The direct stress in the post is nothing else than

$$S = \frac{2P(h-h_0)}{b}.$$

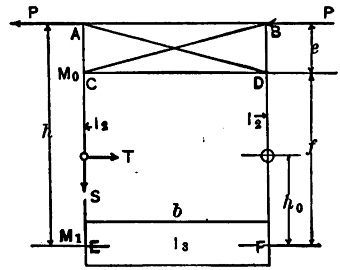

Fig. 91

116. With the bracing of Fig. 91, by referring back to Art. 22, it will at once be seen that here again

$$T = H = \frac{P+P}{2},$$

and taking moments at C, E, and O,

$$M_0 = -T(f - h_0) = -H(f - h_0),$$
$$M_1 = Th_0 = Hh_0,$$
$$-Sb - 2P(h - h_0) = 0.$$

Then at any point distant x from E we have the following moments in the post and floor-beam:

$$E \text{ to } C \quad . \quad . \quad M_1 + \frac{M_0 - M_1}{f} x = H\,(h_0 - x),$$

$$C \text{ to } A \quad . \quad . \quad M_0 - \frac{M_0}{e}(x - f) = H\,(f - h_0)\left(\frac{x - f}{e} - 1\right),$$

$$E \text{ to } F \quad . \quad . \quad M_1 - \frac{2\,M_1}{b} x = Hh_0\left(1 - \frac{2\,x}{b}\right).$$

Neglecting the influence of all direct stresses, the total internal work may now be written,

$$\omega = \frac{H^2}{EI_2}\left\{\int_0^f (h_0 - x)^2\,dx + \int_f^h (f - h_0)^2\left(\frac{x - h^2}{e}\right)dx\right\}$$
$$+ \frac{H^2h_0^2}{2\,EI_3}\int_0^b \left(1 - \frac{2\,x}{b}\right)^2 dx.$$

Integrating, we get,

$$\omega = \frac{H^2}{EI_2}\left\{f\left(h_0^2 - h_0 f + \frac{f^2}{3}\right) + \frac{e}{3}(f - h_0)^2\right\} + \frac{H^2h_0^2 b}{6EI_3}\,.$$

Setting the first derivative of ω with respect to h_0 equal to zero, we get,

$$h_0 = \frac{\dfrac{f}{I_2}(2\,h + f)}{\dfrac{2}{I_2}(h + 2f) + \dfrac{b}{I_3}} \quad . \quad . \quad . \quad . \quad . \quad (238)$$

If we make $I_3 = \infty$ in this, the equation would revert to that of the case of posts with the lower ends fixed — discussed in Art. 22.

EXAMPLES. — In a through bridge 18 feet in width, an upper panel wind pressure amounting to 18,000 lbs. is to be brought down to the lower panel point at an intermediate point in the truss. To calculate the stresses produced in the posts and bracing.

In the case of Fig. 92 from Eq. (237),

$$h_0 = \frac{\dfrac{18}{3600} + \dfrac{3 \times 18.5}{2400}}{\dfrac{18}{3600} + \dfrac{6 \times 18.5}{2400} + \dfrac{18}{12{,}000}} \times 18.5 = 9.9 \text{ ft.},$$

so that

$$M_0 = -9000\,(18.5 - 9.9) = -77{,}400 \text{ ft.-lbs.},$$
$$M_1 = 9000 \times 9.9 = 89{,}100 \text{ ft.-lbs};$$

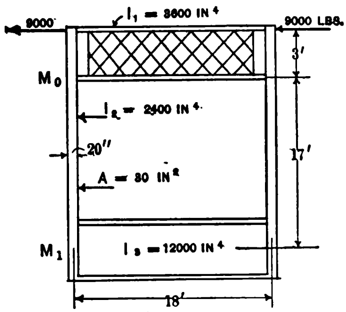

Fig. 92

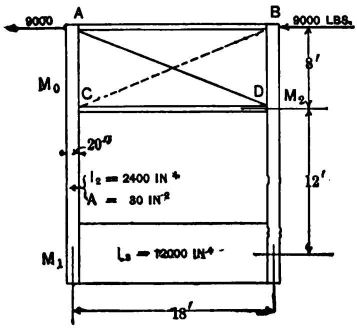

Fig. 93

and since, by using the designation in Fig. 90,

$$-Sb - 2P\,(20 - h_0) = 0,$$
$$S = -\frac{18{,}000\,(20-9.9)}{18} = 10{,}100 \text{ lbs.}$$

The maximum stress in the post will therefore be

$$\frac{M_1}{I_2} \times 10 \pm \frac{S}{A} = \frac{89{,}100 \times 12}{2400} \times 10 + \frac{10{,}100}{30} = 4792 \text{ lbs. per sq. in.}$$

The stress in the upper strut due to bending is

$$\frac{77{,}400 \times 12}{3600} \times 18 = 4644 \text{ lbs. per sq. in.}$$

In the case of Fig. 93, we have, from Eq. (238),

$$h_0 = \frac{\frac{12}{2400}(40 + 12)}{\frac{2}{2400}(20 + 24) + \frac{18}{12{,}000}} = 6.8 \text{ ft.},$$

so that

$$M_0 = -9000(12 - 6.8) = -46{,}800 \text{ ft.-lbs.},$$
$$M_1 = 9000 \times 6.8 = 61{,}200 \text{ ft.-lbs.},$$
$$S = -\frac{18{,}000(20 - 6.8)}{18} = 13{,}200 \text{ lbs.}$$

The maximum stress in the post will therefore be

$$\frac{61{,}200 \times 12}{2400} \times 10 + \frac{13{,}200}{30} = 3500 \text{ lbs. per sq. in.}$$

The stresses in the bracing are obtained in the following manner:

Passing a section through bracings and one of the points of contraflexure, as shown in Fig. 94, and considering the left portion, we have, by taking moment at A,

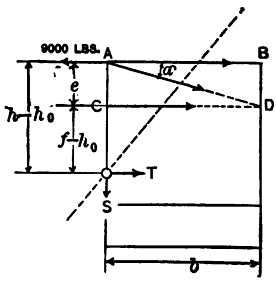

Fig. 94

$$-T(h - h_0) - \overline{CD}e = 0,$$
$$\overline{CD} = -\frac{9000(20 - 6.8)}{8} = -14{,}850 \text{ lbs.}$$

Taking moment at D,

$$-T(f - h_0) - Sb - 9000e + \overline{AB}e = 0,$$
$$-H(f - h_0) + 2H(h - h_0) - 9000e + \overline{AB}e = 0,$$
$$\overline{AB} = -\frac{9000(20 - 6.8)}{8} = -14{,}850 \text{ lbs.}$$

Since at the section Σ vert. forces $= 0$,

$$-S - \overline{AD}\sin\alpha = 0,$$

$$\overline{AD} = \frac{2H(h - h_0)}{b\sin\alpha} = \frac{18{,}000(20 - 6.8)}{18 \times .4062} = 32{,}500 \text{ lbs.}$$

APPENDIX I

A PROOF OF THE THEOREMS OF CASTIGLIANO

THE following proof of the theorems is in its method essentially the same as that given by Professor Föppl in his "Technische Mechanik." Let a number of forces (or moments) P's be simultaneously and each gradually applied on an elastic body, making corresponding displacements (or rotations) y's.

Since the internal work equals the external work, we at once have

$$\omega = \tfrac{1}{2}\Sigma Py,$$

ω denoting the amount of internal work.

Suppose now that one of the forces, say P_v, is a variable one, and receives a uniformly varying increment dP_v, which will be assumed to attain its ultimate amount simultaneously with P_v; then, since each y is a certain function of P_v, we have for the first derivative of any one term Py with respect to P_v,

$$P\frac{dy}{dP_v},$$

and so for all other terms, except $P_v y_v$, as P_v and y_v are both variable quantities.

For this we have

$$\frac{d(P_v y_v)}{dP_v} = \frac{P_v dy_v + y_v dP_v}{dP_v} = P_v\frac{dy_v}{dP_v} + y_v.$$

Summing up the results, we get

$$\frac{d\omega}{dP_v} = \frac{1}{2}\Sigma P \frac{dy}{dP_v} + \frac{1}{2} y_v \quad . \quad . \quad . \quad . \quad . \quad (a)$$

If, on the other hand, we suppose the body in the first place to be acted on by all the P's and to these dP_v to be subsequently added, the ultimate amount of the ω will be the same as in the preceding case. In the present case, the gradual application of dP_v changes y of any one force P by dy with respect to P_v, whereby the force which remains constant performs the work

$$Pdy,$$

and so with other forces including P_v.

The work performed by dP_v itself amounts to

$$\frac{1}{2} dP_v dy_v.$$

As this latter, however, is a quantity of the second order, we may neglect it in comparison with the increment of works performed by the forces themselves, and put

$$\frac{d\omega}{dP_v} = \Sigma P \frac{dy}{dP_v} \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Combining equations (a) and (b) we get

$$\frac{d\omega}{dP_v} = \frac{1}{2}\Sigma P \frac{dy}{dP_v} + \frac{1}{2} y_v = \Sigma P \frac{dy}{dP_v},$$

from which

$$\frac{d\omega}{dP_v} = y_v,$$

which proves the first theorem.

The second theorem is, as already explained, a special case of the first one, and is at once deducible from the latter.

APPENDIX II

TEMPERATURE STRESSES IN VIADUCT BENTS

(Being a supplement to Chap. II)

THE temperature stresses in an ordinary viaduct bent are often not possible of combination with stresses arising from other causes, unless the temperature at the time of erection is exactly known. The following discussion concerns the stresses produced by any given change in temperature alone.

Let

t = the range of a temperature change;

θ = coefficient of expansion and contraction.

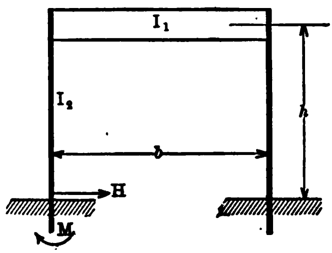

Fig. 1

Then in the bent of Fig. 1 a rise of t above an initial temperature would increase the distance b between the posts by $t\theta b$ were the latter free to move.

However, the lower ends of the posts being assumed to be firmly fixed, moment and horizontal reaction would be produced at each end; each horizontal reaction H acting, as it were, through a distance of

$$\frac{t\theta b}{2}.$$

Calling the moments producing compression on the outside fibre of the frame positive, we have for the bending moment at any point in the post,

$$M - Hx,$$

and in the cross-girder,

$$M - Hh.$$

Then, neglecting the effect of direct stresses, we get for the internal work in the frame,

$$\omega = \frac{1}{EI_2}\int_0^h (M - Hx)^2\,dx + \frac{1}{2\,EI_1}\int_0^b (M - Hh)^2 dx$$

$$= \frac{h}{EI_2}\left(M^2 - HMh + \frac{H^2h^2}{3}\right) + \frac{b}{2\,EI_1}(M - Hh)^2,$$

in which I_1 and I_2 denote the moments of inertia of the cross-girder and posts respectively, and E the modulus of elasticity of the material.

Then, according to the principles of work,

$$\frac{d\omega}{dM} = 0, \qquad \frac{d\omega}{dH} = t\theta b.$$

So that we at once get

$$\frac{h}{I_2}(2\,M - Hh) + \frac{b}{I_1}(M - Hh) = 0,$$

$$\frac{h}{3I_2}(3\,M - 2\,Hh) + \frac{b}{I_1}(M - Hh) = -\frac{Et\theta b}{h},$$

from which

$$H = \frac{2\,hI_1 + bI_2}{h\,(hI_1 + bI_2)}\,M,$$

$$M = \left(\frac{hI_1 + bI_2}{hI_1 + 2\,bI_2}\right)\frac{3\,EI_2t\theta b}{h^2}.$$

In case the lower ends of the posts are hinged, $M = 0$, so that we get

$$H = \left(\frac{3 I_1 I_2}{2hI_1 + 3bI_2}\right) \frac{Et\theta b}{h^2}.$$

Referring to the numerical example given on page 27, for

$$t = 50° \text{ Fah.},$$
$$\theta = .000007,$$
$$E = 30{,}000{,}000 \text{ lbs. per sq. in.},$$

we get

$$M = 263{,}400 \text{ in.-lbs.},$$
$$H = 2840 \text{ lbs.},$$
$$M - Hh = 247{,}800 \text{ in.-lbs.},$$

and for the same with lower ends of posts hinged,

$$H = 707 \text{ lbs.},$$
$$- Hh = 127{,}260 \text{ in.-lbs.},$$

showing that the moment in the post due to temperature change is very much augmented by rigid connections, while against horizontal forces the reverse would be true —a fact perhaps so well known as to hardly require any further comment.

Longitudinally, the temperature stresses produced in the posts also differ considerably according to the modes of connection made between the girder and the posts and the number of spans so connected.

Suppose that in Fig. 2, two continuous spans are fixed longitudinally at C and movable at A and the posts firmly fixed at D and rigidly riveted to the girder.

Then a rise t in temperature will tend to displace the point B with respect to D by $t\theta l$, so that H would be acting, as it were, through that distance in the direction as shown in the figure.

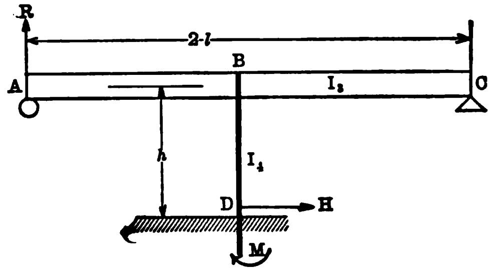

Fig. 2

Calling moments producing compression on the upper fibre of the girder and on the left side one of the post positive, we have for moment at any point of

AB	Rx	with origin of x at A,
BD	$M - Hx$	" " " x at D,
BC	$\frac{Rl + M - Hh}{l}x$	" " " x at C.

Again neglecting the effect of direct stresses, we get for the internal work in the frame,

$$\omega = \frac{1}{2EI_3}\left\{\int_0^l (Rx)^2 dx + \int_0^l \left(\frac{Rl + M - Hh}{l}x\right)^2 dx\right\}$$

$$+ \frac{1}{2EI_4}\int_0^h (M - Hx)^2 dx,$$

in which I_3 and I_4 represent the moments of inertia of the girder and posts respectively, both considered as being constant throughout.

Then since according to the principles of work

$$\frac{d\omega}{dH} = t\theta l, \quad \frac{d\omega}{dM} = 0, \quad \frac{d\omega}{dR} = 0,$$

we get

$$\frac{2l}{I_3}(Hh - M - Rl) + \frac{h}{I_4}(2Hh - 3M) = \frac{6Et\theta l}{h},$$

$$\frac{2l}{I_3}(H - Mh - Rl) + \frac{3h}{I_4}(Hh - 2M) = 0,$$

$$Hh - M - 2Rl = 0,$$

from which

$$M = \left(\frac{lI_4 + 3hI_3}{2lI_4 + 3hI_3}\right)\frac{6EI_4t\theta l}{h^2},$$

$$H = \left(\frac{lI_4 + 6hI_3}{2lI_4 + 3hI_3}\right)\frac{6EI_4t\theta l}{h^3}.$$

If the lower ends of the posts were hinged at D, M would disappear, so that we get

$$H = \left(\frac{I_3I_4}{lI_4 + 2hI_3}\right)\frac{6Et\theta l}{h^2}.$$

Again referring to the previous example and assuming

$I_3 =$	30,000 in.^{4}	$l =$	50 ft.
$I_4 =$	1000 in.^{4}	$h =$	15 ft.

we get for the case of fixed posts,

$$M = 1{,}126{,}400 \text{ in.-lbs.},$$

$$H = 12{,}290 \text{ lbs.},$$

$$M - Hh = 1{,}085{,}800 \text{ in.-lbs.}$$

A comparison of such figures will show that in most cases occurring in practice, the max. moment in the post when the latter is firmly connected to the girder and to the foundation may, without material error, be assumed to be twice that produced when either end is hinged.

The combined action of lateral and longitudinal moments is to throw the greatest compression, in the above case, on the outermost corner on the left side of the post base, where it would amount to, for a 12-in. square post with $I = 1000$ in.^{4} in either direction:

$$\frac{263{,}400}{1000} \times 6 + \frac{1{,}126{,}400}{1000} \times 6 = 8340 \text{ lbs. per sq. in.}$$

By applying the foregoing computations to any form of post sections, it will at once be seen that a considerable stress is produced in certain portions of the column, in the kind of bent discussed, under changes of temperature which are not uncommon.

JULY, 1917

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